We wish the prove a continuous real-valued function on a closed interval in $\mathbb{R}$ is integrable. Without loss of generality, let the interval be $[a,b] \in \mathbb{R}$ for $a < b$. We know that a real-valued function f on $[a,b]$ is integrable if and only of for any $\epsilon > 0$, there exists $\delta > 0$ such that $|S_1 - S_2| < \epsilon$ whenever S_1, S_2 are Riemann sums for f corresponding to partitions of $[a,b]$ of mesh smaller than δ.

Let P_1, P_2 be partitions of the interval $[a,b]$ correspondoing to Riemann sums S_1, S_2 respectively, with $mesh(P_1), mesh(P_2) < \delta$. Let $P = P_1 \cup P_2$ be the refinement of the two partitions. Clearly $mesh(P) < \delta$, defined as

$$P = \{a = x_0 < x_1 < \cdots < x_N\}.$$

Re-define the tags T_1, T_2 for the Riemann integrals S_1, S_2. By the definition of Riemann integrals,

$$S_1 = \sum_{i=1}^{N} f(t_{1i})(x_i - x_{i-1}) \quad t_i \in [x_{i-1}, x_i]$$

$$S_2 = \sum_{i=1}^{N} f(t_{2i})(x_i - x_{i-1}) \quad t_i \in [x_{i-1}, x_i]$$

Then we have

$$|S_1 - S_2| = \sum_{i=1}^{N} |f(t_{1i}) - f(t_{2i})|\,(x_i - x_{i-1}) \quad t_i \in [x_{i-1}, x_i]$$

However, since f is continuous over a compact set $[a,b]$, f uniformly continuous, meaning there exists $\delta > 0$ such that if $x, y \in [a,b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. Since we have $mesh(P) < \delta$,

$$|S_1 - S_2| = \sum_{i=1}^{N} |f(t_{1i}) - f(t_{2i})|\,(x_i - x_{i-1}) < \sum_{i=1}^{N} \frac{\epsilon}{b-a}(x_i - x_{i-1}) = \epsilon\frac{b-a}{b-a} = \epsilon$$

Then by Lemma 1 of section 3, a continuous real-valued function on a closed interval in $\mathbb{R}$ is integrable.

13. Note that $f(x)$ attains its max M at some point d on the interval $[a,b]$ since the interval is compact and $f(x)$ is continuous. Since x^n is a monotonically increasing function for $x > 0$ and $n > 0$, $M^n \geq f^n(x) \forall x \in [a,b]$. Since $f(x)$ is also nonnegative for $x \in [a,b]$, we have that $\int_a^b M^n dx \geq \int_a^b f^n(x)dx$, so $M(b-a)^{\frac{1}{n}} \geq (\int_a^b f^n(x)dx)^{\frac{1}{n}}$. Now note that since $f(x)$ is continuous, $\exists \delta > 0$ such that $f(x) > (M - \epsilon) \forall x \in [d-\delta, d+\delta] \cap [a,b]$ for any $\epsilon > 0$. Define $g(x) = M - \epsilon$ on $x \in [d-\delta, d+\delta] \cap [a,b]$ and $g(x) = 0$ everywhere else on $[a,b]$. Then $f^n(x) \geq g^n(x) \geq 0 \forall x \in [a,b]$. Hence $\int_a^b f^n(x)dx \geq \int_a^b g^n(x)dx \geq (M-\epsilon)^n \delta$ and $(\int_a^b f^n(x)dx)^{\frac{1}{n}} \geq (M-\epsilon)\delta^{\frac{1}{n}}$. We now have $M(b-a)^{\frac{1}{n}} \geq (\int_a^b f^n(x)dx)^{\frac{1}{n}} \geq (M-\epsilon)\delta^{\frac{1}{n}}$, so $\lim_{n\to\infty} M(b-a)^{\frac{1}{n}} \geq \lim_{n\to\infty} (\int_a^b f^n(x)dx)^{\frac{1}{n}} \geq \lim_{n\to\infty} (M-\epsilon)\delta^{\frac{1}{n}}$. But since $b - a > 0$, $\lim_{n\to\infty} M(b-a)^{\frac{1}{n}} = M$ and since $\delta > 0$, $\lim_{n\to\infty} (M-\epsilon)\delta^{\frac{1}{n}} = M - \epsilon$. Hence $M \geq \lim_{n\to\infty} (\int_a^b f^n(x)dx)^{\frac{1}{n}} \geq M - \epsilon$. Letting ϵ go to 0, we get the desired result $\lim_{n\to\infty} (\int_a^b f^n(x)dx)^{\frac{1}{n}} = M$.

14

We want to prove that if f is a continuous real-valued function on $\{x \in \mathbb{R} : x \geq 0\}$ and $\lim_{x\to+\infty} f(x) = c$,

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^x f(t)dt = c.$$

This is equivalent to showing that given any $\epsilon > 0$, there exists $N > 0$ such that if $x > N$ then

$$\left|\frac{1}{x}\int_0^x f(t)dt - c\right| < \epsilon.$$

We start by noticing that since f is continuous on $[0, x]$, by the Intermediate Value Theorem for integrals, there exists $b \in [0, x]$ such that

$$f(b) = \frac{1}{x}\int_0^x f(t)dt.$$

Then for $c \in [0, x]$, we have

$$\left|\frac{1}{x}\int_0^x f(t)dt - c\right| = |f(b) - c|$$

We know that $\lim_{b\to+\infty} f(b) = c$, so given any $\epsilon > 0$ for there exists some $N > 0$ such that

$$\left|\frac{1}{x}\int_0^x f(t)dt - c\right| = |f(b) - c| < \epsilon$$

whenever $b > N$. However, since $b \in [0, x]$, $b \leq x$. Therefore, $b > N$ implies $x > N$. Therefore, we found an N such that given any $\epsilon > 0$, if $x > N$ then

$$\left|\frac{1}{x}\int_0^x f(t)dt - c\right| < \epsilon.$$

15

Let $D = [a, b] \times [c, d]$, and let $g(y) = \int_a^b f(x, y)dx$. We want to show that given any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y_0 \in [c, d]$, if $y \in [c, d]$ then $|g(y) - g(y_0)| < \epsilon$ whenever $|y - y_0| < \delta$.

$$\begin{aligned}|g(y) - g(y_0)| &= \left|\int_a^b f(x, y)dx - \int_a^b f(x, y_0)dx\right| = \left|\int_a^b (f(x, y) - f(x, y_0))dx\right| \\ &\leq \int_a^b |f(x, y) - f(x, y_0)|\, dx\end{aligned}$$

Since f is a continous real-valued function on D, there exists $\delta_1 > 0$ such that $f(x_a, y_a) - f(x_b, y_b) < \epsilon$ whenever $d((x_a, y_a), (x_b, y_b)) = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2} < \delta_1$. In particular, there exists $\delta > 0$ such that $f(x, y) - f(x, y_0) < \frac{\epsilon}{b-a}$ whenever $d((x, y), (x, y_0)) = |y - y_0| < \delta$. In fact, since D is compact, f is uniformly continuous so we can find a $\delta > 0$ such that this holds for all $y_0 \in [c, d]$. Then we have

$$|g(y) - g(y_0)| \leq \int_a^b |f(x, y) - f(x, y_0)|\, dx < \int_a^b \frac{\epsilon}{b-a}dx = \epsilon\frac{b-a}{b-a} = \epsilon.$$

This proves that $g(y) = \int_a^b f(x,y)dx$ is continuous for all $y \in [c,d]$.

16

Let $f, g \in C([a,b])$. $C([a,b])$ is a metric space with the metric $\max\{|f_1(x) - f_2(x)| : x \in [a,b]\}$ for $f_1, f_2 \in C([a,b])$. We want to show that given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \int_a^b f(t)dt - \int_a^b g(t)dt \right| < \epsilon$$

whenever $\max\{|f(x) - g(x)| : x \in [a,b]\} < \delta$.

$$\begin{aligned} \left| \int_a^b f(t)dt - \int_a^b g(t)dt \right| &= \left| \int_a^b (f(t) - g(t))\, dt \right| \leq \int_a^b |f(t) - g(t)|\, dt \\ &< \int_a^b \delta dt = (b-a)\delta \end{aligned}$$

Choose $\delta = \frac{\epsilon}{b-a}$. Then we have

$$\left| \int_a^b f(t)dt - \int_a^b g(t)dt \right| < (b-a)\delta = (b-a)\frac{\epsilon}{b-a} = \epsilon.$$

1.

Proof. Since f is continuous on the closed set $[a,b]$, f is bounded and attains its max and min, ie. $f(t_0) \leq f(t) \leq f(t_1)$ for all $t \in [a,b]$ where $f(t_0)$ is the min and $f(t_1)$ is the max. Since g is nonnegative for all values $t \in [a,b]$, we know that

$$f(t_0)g(t) \leq f(t)g(t) \leq f(t_1)g(t)$$

By integrating both sides we get

$$f(t_0)\int_a^b g(t)dt \leq \int_a^b f(t)g(t)dt \leq f(t_1)\int_a^b g(t)dt$$

If $\int_a^b g(t)dt = 0$, then $g(t) = 0$ and the result is trivially true. Therefore, assume $g(t) \neq 0$. Then

$$f(t_0) \leq \frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt} \leq f(t_1)$$

so by the Intermediate Value Theorem, there exists an $x \in [a,b]$ such that

$$f(x) = \frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt} \implies \int_a^b f(t)g(t)dt = f(x)\int_a^b g(t)dt$$

as desired.

To prove the Mean Value Theorem, take $g(t) = 1$. We know from class that $\int_a^b dt = (b-a)$. It then imediately follows from above that

$$\int_a^b f(t) \cdot 1dt = f(x)\int_a^b 1dt = f(x)(b-a)$$

Therefore, the Mean Value Theorem holds for integrals.

□