Let $b_k = a_{k+m}$, for positive integers k and m. We need to prove that $\sum_{k=1}^{\infty} b_k$ if and only if $\sum_{k=1}^{\infty} a_k$ converges.

Let $\sum_{k=1}^{\infty} b_k$ converge. This means that for some $S \in \mathbb{R}$, given any $\epsilon > 0$ there exists $N > 0$ such that for any $n > N$ we have

$$\left|\sum_{k=1}^{n} b_k - S\right| < \epsilon.$$

Now consider $\sum a_k$. For $n > m$,

$$\left|\sum_{k=1}^{n} a_k - (a_1 + a_2 + \cdots + a_m + S)\right| = \left|a_1 + \cdots + a_m + \sum_{k=m+1}^{n} a_k - (a_1 + a_2 + \cdots + a_m + S)\right|$$

$$= \left|\sum_{k=m+1}^{n} a_k - S\right| = \left|\sum_{k=1}^{n-m} a_{k+m} - S\right| = \left|\sum_{k=1}^{n-m} b_k - S\right| < \epsilon$$

Since $\sum_{k=1}^{\infty} b_k$ converges to S and m is a fixed number such that $n > m$, given any $\epsilon > 0$ we can find N such that for all $n - m > N$ (or equivalently $n > m + N$) such that the above inequality holds. Thus, $\sum a_k$ converges, and we have

$$\sum_{k=1}^{\infty} a_k = a_1 + \cdots + a_m + \sum_{k=1}^{\infty} b_k = a_1 + \cdots + a_m + \sum_{k=1}^{\infty} a_{k+m}$$

Now let $\sum_{k=1}^{\infty} a_k$ converge. We wish to show that given any $\epsilon > 0$, there exists postive integer N such that for $p > q \geq N$

$$\left|\sum_{k=q}^{p} b_k\right| < \epsilon.$$

$$\left|\sum_{k=q}^{p} b_k\right| = \left|\sum_{k=q}^{p} a_{k+m}\right| = \left|\sum_{k=q+m}^{p+m} a_k\right|$$

But since $\sum_{k=1}^{\infty} a_k$ converges, by Cauchy criterion there exists positive integer N such that for all $p + m > q + m \geq N$, or equivalently $p > m \geq N - m$ we have

$$\left|\sum_{k=q}^{p} b_k\right| = \left|\sum_{k=q+m}^{p+m} a_k\right| < \epsilon.$$

By Cauchy criterion, $\sum_{k=1}^{\infty} b_k$ converges.

Problem 8

Let $a_1, a_2, a_3, ...$ be a decreasing sequence of positive numbers.

a) $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$ converges so for $\varepsilon > 0$, let $s_n = \sum_{k=1}^{n} a_k$ and since $\{s_n\}$ converges, $\{s_n\}$ is Cauchy and $\exists N$ such that for $n, m \geq N \Rightarrow |s_n - s_m| < \varepsilon$. Thus:

$$(n-N)a_n \leq a_{N+1} + ... + a_n = |s_n - s_N| < \varepsilon \Rightarrow \lim_{n\to\infty} (n-N)a_n = 0$$

Since $\lim_{n\to\infty} Na_n = N \lim_{n\to\infty} a_n = 0$

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} (n-N)a_n + \lim_{n\to\infty} Na_n = 0$$

b) $a_1 + a_2 + a_3 + ...$ converges iff $a_1 + 2a_2 + 4a_4 + 8a_8 + ...$ converges.

Let $\sum_{n=1}^{\infty} a_n$ converge

Let t_n be a sequence of partial sums with $t_n = \sum_{k=1}^{n} 2^{k-1} a_{2^{k-1}} = a_1 + 2a_2 + 4a_4 + 8a_8 + ... + 2^{n-1}a_{2^{n-1}}$

$$t_n = a_1 + 2a_2 + 4a_4 + 8a_8 + ... + 2^{n-1}a_{2^{n-1}} \leq 2(\frac{1}{2}a_1 + a_2 + 2a_4 + ... + 2^{n-2}a_{2^{n-1}})$$

$$\leq 2[a_1 + a_2 + (a_3 + a_4) + ... + (a_{2^{n-2}+1} + ... + a_{2^{n-1}})] \leq 2\sum_{n=1}^{\infty} a_n$$

t_n is bounded above so by the comparison test, it converges.

Let $a_1 + 2a_2 + 4a_4 + 8a_8 + ...$ converge:

a_n is a decreasing sequence of positive numbers so $a_{n+1} \leq a_n$ so $|a_2 + a_3| \leq a_2 + a_2 = 2a_2$ and $|a_4 + a_5 + a_6 + a_7| \leq a_4 + a_4 + a_4 + a_4 = 4a_4$ and so on. Thus for all a_n, $\sum_{n=1}^{\infty} a_n \leq a_1 + 2a_2 + 4a_4 + 8a_8 + ...$ and by the comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

9

Let $\sum_{k=1}^{\infty} f(k)$ converge to a positive real number S. By drawing a picture, it is clear that

$$\int_k^{k+1} f(x)dx \leq f(k).$$

Then we have

$$0 < \int_1^{n+1} f(x)dx \leq \sum_{k=1}^{n} f(k)$$

Then we have

$$0 < \lim_{n\to\infty} \int_1^{n+1} f(x)dx \leq \sum_{k=1}^{\infty} f(k) = S.$$

Since the sequence $S_n = \int_1^{n+1} f(x)dx$ is monotonically increasing (since f is positive) and is bounded above, the sequence converges. Hence, $\lim_{n\to\infty} \int_1^{n+1} f(x)dx$ exists.

Now let $\lim_{n\to\infty} \int_1^{n} f(x)dx$ exist and converge to a positive real number S. By drawing a picture, it is easy to see that

$$f(k+1) \leq \int_k^{k+1} f(x)dx.$$

Therefore,

$$\sum_{k=2}^{n} f(k) \leq \int_1^{n+1} f(x)dx.$$

$$0 < \sum_{k=2}^{\infty} f(k) \leq \lim_{n\to\infty} \int_1^{n} f(x)dx = S.$$

Since the sequence $S_n = \sum_{k=2}^{n} f(k)$, $n = 2, 3, \cdots$ is monotonically increasing and is bounded above, the sequence converges. Since f is defined at $x = 1$, $f(1)$ is a finite real number. Therefore,

$$f(1) + \sum_{k=2}^{\infty} f(k) = \sum_{k=1}^{\infty} f(k)$$

converges.

10. Use the preceding problem to tell for which $p > 0$ the following series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \sum_{n=3}^{\infty} \frac{1}{n\log n(\log \log n)^p}.$$

(a) $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Let $f(n) = \frac{1}{n^p}$, then $f(x)$ is decreasing, positive and continuous for $x \geq 1$.
By integral test,

$$f(x) = \begin{cases} \int_1^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x} dx = \lim_{b\to\infty} [\ln x]_1^b = \infty & \text{if } p = 1 \\ \int_1^{\infty} x^{-p} = \lim_{b\to\infty} \int_1^b x^{-p} dx = \lim_{b\to\infty} [\frac{1}{1-p} x^{1-p}]_1^b & \text{if } p \neq 1 \end{cases}$$

$(p \neq 1)$, $\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b\to\infty} \frac{1}{1-p} b^{(1-p)} - \frac{1}{1-p}$.

If $1 - p > 0$, the improper integral diverges since $\lim_{n\to\infty} b^{(1-p)} = \infty$.
and if $1 - p < 0$, the improper integral converges since $\lim_{n\to\infty} b^{(1-p)} = 0$.

Therefore by integral test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverge if $p \leq 1$.

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$

Let $f(n) = \frac{1}{n(\log n)^p}$. Compute $\int_2^{\infty} f(x)dx$. Let's make the substitution, $u = \log x$, then $du = \frac{1}{x} dx$ so our integral becomes $\int \frac{1}{xu^p} x du = \int u^{-p} du$
If $p = 1$, then $\int_2^{\infty} f(x)dx = \log u = \log(\log x)|_2^{\infty} = \lim_{b\to\infty} \log(\log b) - \log(\log 2) = \infty$. Thus, it diverges when $p = 1$.
If $p \neq 1$, then $\int_2^{\infty} f(x)dx = \frac{1}{1-p} u^{-p+1} = \frac{1}{1-p} (\log x)|_2^{\infty} = \lim_{b\to\infty} \frac{1}{1-p} [(\log b)^{-p+1} - (\log 2)^{-p+1}]$.
We know that it converges if $-p + 1 < 0$, and diverges if $-p + 1 > 0$. Therefore the series converges if $p > 1$ and diverges if $p \leq 1$.

(c) $\sum_{n=3}^{\infty} \frac{1}{n\log n(\log \log n)^p}$.

Likewise (b), Let's make the substitution, $u = \log(\log x)$, then $du = \frac{1}{x\log x} dx$ so our integral becomes $\int u^{-p} du$
If $p = 1$, $\int_3^{\infty} f(x)dx = \log u = \log[\log(\log x)]|_3^{\infty} = \infty$
If $p \neq 1$, then $\int_3^{\infty} f(x)dx = \frac{1}{1-p} [\log(\log x)]^{-p+1}|_3^{\infty} = \lim_{b\to\infty} \frac{1}{1-p} [\log(\log b)]^{-p+1} - [\log(\log 3)]^{-p+1}$
. Likewise (b), we know that it converges if $-p+1 < 0$, and diverges if $-p+1 > 0$. Therefore the series converges if $p > 1$ and diverges if $p \leq 1$

14)

Let $\{a_n\}$ be a conditionally convergent series, and partition its elements into the negatives and positives. Observe that $\sum_{a_n>0} a_n$ and $\sum_{a_n<0} a_n$ both diverge, for if they converged, then $\sum_{a_n>0} a_n - \sum_{a_n<0} a_n = \sum |a_n|$ converges, contrary to the assumption.

Let a be an arbitrary value that wish for our rearrangement to converge to. Without loss of generality, assume $a > 0$, then choose from the set of $\{a_n : a_n > 0\}$ such that the sum is greater than a. Then add from the set of $\{a_n : a_n < 0\}$ so that the sum is less than a, and alternate between the sets so we are choosing just enough we switch and the sum is greater than or less than a. Denote the nth iterate of this process by the partial sum $P_1 + N_1 + P_2 + N_2 + \cdots + P_n + N_n$, where P's denote we choose from a_n^+ so the partial sum exceeds a, and N's denote we choose from a_n^- until the partial sum is less than a. Observe

$$P_1 + N_1 + P_2 + N_2 + \cdots + P_n + N_n \leq a \leq P_1 + N_1 + P_2 + N_2 + \cdots + P_n + N_n + \bar{a}$$

$$\Rightarrow P_1 + N_1 + P_2 + N_2 + \cdots + P_n + N_n - a \leq \bar{a}$$

Where $\bar{a}$ denotes the next positive element of $\{a_n\}$ note yet used in our partial sum. We also have

$$a - (P_1 + N_1 + P_2 + N_2 + \cdots + P_n + N_n) \leq \bar{a}$$

$$\Rightarrow -(P_1 + N_1 + P_2 + N_2 + \cdots + P_n + N_n - a) \geq -\bar{a}$$

So $|P_1 + N_1 + P_2 + N_2 + \cdots + P_n + N_n - a| \leq \bar{a}$. Since $\{a_n\}$ is a conditionally convergent series, $a_n \to 0$, so the subsequence of positive terms also converge to 0. Thus for $\epsilon > 0$, we choose N such that $\bar{a} < \epsilon$, then we will have $|P_1 + N_1 + P_2 + N_2 + \cdots + P_n + N_n - a| \leq \bar{a} < \epsilon$, as desired.

15) Prove that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent series of real numbers then the series $\sum_{m,n=1}^{\infty} a_n b_m$ is also absolutely convergent, and

$$\sum_{n,m=1}^{\infty} a_n b_m = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{m=1}^{\infty} b_m\right)$$

For $\sum_{n=1}^{\infty} a_n$ to converge, $|a_n| \leq M$ for some M and all n. Thus $|a_n b_m| \leq |M b_m|$ for all n. Since $\sum_{n=1}^{\infty} b_n$ converges absolutely, so does $\sum_{n=1}^{\infty} M b_n$. By comparison, $\sum_{m,n=1}^{\infty} a_n b_m$ also converges absolutely. To show the identity:

$$\sum_{n,m=1}^{\infty} a_n b_m = \lim_{k\to\infty}\left(\sum_{m=1}^{\infty} a_1 b_m + \sum_{m=1}^{\infty} a_2 b_m + \cdots + \sum_{m=1}^{\infty} a_k b_m\right)$$

$$= \lim_{k\to\infty}\left(a_1 \sum_{m=1}^{\infty} b_m + a_2 \sum_{m=1}^{\infty} b_m + \cdots + a_k \sum_{m=1}^{\infty} b_m\right)$$

$$= \left(\sum_{m=1}^{\infty} b_m\right) \lim_{k\to\infty}[(a_1 + a_2 + \cdots + a_k)] = \left(\sum_{m=1}^{\infty} b_m\right) \lim_{k\to\infty} \sum_{n=1}^{k} a_1 = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{m=1}^{\infty} b_m\right)$$

Additional Problem 1

Let u and v be continuous functions on $[a, b]$ and differentiable on (a, b) and let u' and v' be integrable on $[a, b]$. u and v are differentiable and continuous so uv is differentiable on (a, b) and u and v are integrable on $[a, b]$. $(uv)' = (uv' + u'v)$ and since the product and sums of integrable functions are integrable, $\int_a^b (uv' + u'v)$ exists.

$$\int_a^b (uv)'(x) = \int_a^b (u(x)v'(x) + u'(x)v(x))dx = \int_a^b u(x)v'(x)dx + \int_a^b u'(x)v(x)dx$$

By the fundamental theorem of integral calculus:

$$\int_a^b (u(x)v(x))' = u(b)v(b) - u(a)v(a) = \int_a^b u(x)v'(x)dx + \int_a^b u'(x)v(x)dx$$

Additional Problem 2

Let $\sum a_n$ and $\sum b_n$ be convergent series of non-negative numbers. We want to show that $\sum \sqrt{a_n b_n}$ converges.

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$\sum a_n$ and $\sum b_n$ converges so given any $\varepsilon > 0$, $\exists N_a$ such that if $n > m \geq N_a$ then $|\sum_m^n a_k| < \varepsilon/2$ and $\exists N_b$ such that if $n > m \geq N_b$ then $|\sum_m^n b_k| < \varepsilon/2$

$$a_n b_n \leq a_n^2 + 2a_n b_n + b_n^2 = (a_n + b_n)^2 \Rightarrow \sqrt{a_n b_n} \leq \sqrt{(a_n + b_n)^2}$$

$$\Rightarrow \sqrt{a_n b_n} \leq \sqrt{(a_n + b_n)^2} = |a_n + b_n| = |a_n| + |b_n|$$

$$\Rightarrow \sum_m^n \sqrt{a_k b_k} \leq |\sum_m^n a_k| + |\sum_m^n b_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus given any $\varepsilon > 0$, $\exists N$ such that if $n > m \geq N$ then $|\sum_m^n \sqrt{a_k b_k}| < \varepsilon$ so $\sum \sqrt{a_n b_n}$ converges.

Additional Problem 3

Let $\sum a_n$ be a convergent series of non-negative numbers and since it converges, $\lim_{n \to \infty} a_n = 0$. Thus, there exists an integer N such that for $n \geq N$, $0 \leq a_n < 1$. And for $p > 1$, for each $n \geq N$, $|a_n^p| \leq a_n$. Then, applying the comparison test, we see that since $\sum a_n$ converges, so does $\sum a_n^p$.