1. Find a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to 0}\lim_{n\to\infty} f_n(x)$ and $\lim_{n\to\infty}\lim_{x\to 0} f_n(x)$ exists and are unequal.

Proof:

$$f_n(x) = \frac{nx}{1+nx}$$

Then,

$$\begin{aligned}\lim_{x\to 0}\lim_{n\to\infty} f_n(x) &= \lim_{x\to 0}\lim_{n\to\infty} \frac{nx}{1+nx} \\ &= \lim_{x\to 0}\lim_{n\to\infty} \frac{nx/nx}{(1+nx)/nx} \\ &= \lim_{x\to 0}\lim_{n\to\infty} \frac{1}{1+\frac{1}{nx}} \\ &= \lim_{x\to 0} \frac{1}{1+0} = 1.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{n\to\infty}\lim_{x\to 0} f_n(x) &= \lim_{n\to\infty}\lim_{x\to 0} \frac{nx}{1+nx} \\ &= \lim_{n\to\infty} \frac{n\cdot 0}{n\cdot 0+1} \\ &= \lim_{n\to\infty} 0 = 0.\end{aligned}$$

2. If $f : E^2 - \{(0,0)\} \to \mathbb{R}$, three limits we can consider are $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$, $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$, and $\lim_{(x,y)\to(0,0)} f(x,y)$. Compute these limits, if they exists, for $f(x,y) = \frac{xy}{x^2+y^2}$ and for $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$.

Proof:

(a) for $f(x,y) = \frac{xy}{x^2+y^2}$.
$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$, so it is exists.
$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} \frac{0}{x^2} = 0$,so it is exists.
However, $\lim_{(x,y)\to(0,0)} f(x,y)$, Let $x = y$. $\lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2}$.
Let $y = 0$. $\lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$.
Since we have different limits from different directions, the limit does not exits.

(b) for $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$.
$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$,

$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$
$\lim_{(x,y)\to(0,0)} f(x,y)$, Let $x = y$. $\lim_{(x,x)\to(0,0)} \frac{0}{2x^2} = 0$.
Let $y = 0$. $\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$.
Likewise (a), we have different limits from different directions, so the limit does not exits.

3. Find a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ that converges to the zero function and such that the sequence $\int_0^1 f_1(x)\,dx$, $\int_0^1 f_2(x)\,dx$, $\int_0^1 f_3(x)\,dx$, ... increases without bound.

Solution:
Let f_n be defined as

$$f_n(x) = \begin{cases} 4n^3x & 0 \le x \le 1/2n \\ 4n^2 - 4n^3x & 1/2n < x \le 1/n \\ 0 & 1/n < x \le 1 \end{cases}$$

Similar to other problems we've seen in class we can show that for any $\epsilon > 0$, there exists $N(\epsilon, x)$ such that whenever $n > N$ we get, $|f_n(x)| < \epsilon$. But the integral is given by

$$\int_0^1 f_n(x)\,dx = \int_0^{1/2n} 4n^3x\,dx + \int_{1/2n}^{1/n} (4n^2 - 4n^3x)\,dx = n$$

hence, as n increases the sequence $\int_0^1 f_1(x)\,dx$, $\int_0^1 f_2(x)\,dx$, $\int_0^1 f_3(x)\,dx$, ... increases without bound.

4) Find a uniformly convergent sequence of differentiable functions $f_n: (0,1) \to \mathbb{R}$ such that the sequence f_1', f_2', f_3', ... does not converge.

Consider $f_n(x) = \frac{\sin(nx)}{n}$. To see that f_n converges uniformly, for any $\epsilon > 0$, choose $N > 0$ such that if $n > N$, then $\frac{1}{n} < \epsilon$. Then we will have $\left|\frac{\sin(nx)}{n} - 0\right| < \left|\frac{1}{n}\right| < \epsilon$, and that $f_n \to 0$. Each f_n is differentiable since $\sin(nx)$ is differentiable everywhere, and we have $f_n'(x) = \cos(nx)$. Clearly this does not converge to 0.

20. Find the radii of convergence of the following power series:

(a) $\sum_{n=1}^{\infty} n(\log n)x^n$: Use Root and Ratio test
Let $a_n = n(\log n)$, we need to show that $\limsup|a_n|^{\frac{1}{n}} \le \limsup|\frac{a_{n+1}}{a_n}|$.
Then,

$$\begin{aligned}\limsup|a_n|^{\frac{1}{n}} &\le \limsup|\frac{a_{n+1}}{a_n}| \\ &= \limsup|\frac{(n+1)\log(n+1)}{n\log n}| \\ &= \limsup|\frac{n\log(n+1)}{n\log n} + \frac{\log(n+1)}{n\log n}| \\ &= \limsup|\frac{\log(n+1)}{\log n}| + \limsup|\frac{\log(n+1)}{n\log n}| \\ &= 1 + 0 = 1.\end{aligned}$$

Therefore, the radius of convergence is 1.

(b) $\sum_{n=1}^{\infty}(\log n)^{\log n}x^n$: Use exponential trick and root test.

Let $a_n = (\log n)^{\log n}$.
Then,

$$\begin{aligned}
\lim |a_n^{\frac{1}{n}}| &= \lim \exp(\log {a_n}^{1/n}) \\
&= \lim \exp[\log\{(\log n)^{\frac{1}{n}\log n}\}] \\
&= \exp[\lim_{n\to\infty} \frac{\frac{1}{n}log(logn) + \frac{1}{nlogn}logn}{1}] \qquad \text{(L'Hopstal's rule)} \\
&= \exp[\lim_{n\to\infty} \frac{log(logn)+1}{n}] \\
&= \exp[\lim_{n\to\infty} \frac{1}{nlogn}] \\
&= \exp 0 = 1.
\end{aligned}$$

Thus, the radius of convegence is 1.

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}}$: Use Root and Ratio test
Let $a_n = \frac{1}{n\sqrt{n}}$. Then,

$$\begin{aligned}
\lim_{n\to\infty} |\frac{1}{n\sqrt{n}}|^{\frac{1}{n}} &= \lim_{n\to\infty} |n^{\frac{-1}{\sqrt{n}}}| \\
&= \lim_{n\to\infty} \exp[\log(n)^{\frac{-1}{\sqrt{n}}}] \\
&= \lim_{n\to\infty} \exp[\frac{-1}{\sqrt{n}}\log n] \\
&= \exp \lim_{n\to\infty} \frac{-\log n}{\sqrt{n}} \\
&= \exp \lim_{n\to\infty} \frac{-1/n}{\frac{1}{2}n^{\frac{-1}{2}}} \qquad \text{(L'Hospital's rule)} \\
&= \exp \lim_{n\to\infty} \frac{-2}{\sqrt{n}} \\
&= \exp 0 = 1.
\end{aligned}$$

Thus, the radius of convergence is 1.

(d) $\sum_{n=1}^{\infty} \frac{x^n}{(\sqrt{n})^n}$: Use Root test

$$\lim_{n\to\infty} \frac{x^n}{(\sqrt{n})^n} = \lim_{n\to\infty} (\frac{x^n}{(\sqrt{n})^n})^{\frac{1}{n}} = \lim_{n\to\infty} \frac{x}{\sqrt{n}} = |\frac{x}{\sqrt{n}}| \to 0$$

Therefore, the radius of convergence is ∞.

(e) $\sum_{n=1}^{\infty} \frac{n^n x^x}{n!}$: Use Ratio test

$$\limsup |\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}| = \limsup |\frac{(n+1)^n}{n^n}| = \limsup |(1+\frac{1}{n})^n| = e$$

Therefore, the radius of convergenve has to be e .

35. Let $[a, b]$ and $[c, d]$ be closed intervals in R and let f be a continuous real-valued function on $\{(x, y) \in E^2 : x \in [a, b], y \in [c, d]\}$. By Prob. 15, Chap. VI, $\int_a^b f(x, y)dx$ is continuous in y and $\int_a^b f(x, y)dy$ is continuous in x, so that

$$\int_c^d \left(\int_a^b f(x, y)dx \right) dy \text{ and } \int_a^b \left(\int_c^d f(x, y)dy \right) dx$$

exist. Prove that these integrals are equal by computing d/dt of

$$\int_c^d \left(\int_a^t f(x, y)dx \right) dy \text{ and } \int_a^t \left(\int_c^d f(x, y)dy \right) dx$$

for $t \in (a, b)$. Since f is a continuous real-valued function, we can apply the Fundemental Theorem of Calculus to get

$$\frac{d}{dt} \int_a^t \int_c^d f(x, y)dydx = \int_c^d f(t, y)dy$$

. We also know that

$$\frac{d}{dt} \int_c^d \left(\int_a^t f(x, y)dx \right) dy = \int_c^d f(t, y)dy$$

by the theorem on Page 159 of the book. . Since these two derivitives are the same, we can conclude that are functions are equal up to some constant, meaning

$$\int_c^d \left(\int_a^t f(x, y)dx \right) dy = C + \int_a^t \int_c^d f(x, y)dydx$$

. If we set $t = a$, we get that $C = 0$ and if we take $t = b$ we get

$$\int_c^d \left(\int_a^b f(x, y)dx \right) dy = \int_a^b \left(\int_c^d f(x, y)dy \right) dx$$

.

Additional Problem 1:

To show that f is integrable, we attempt to find two step functions f_1, f_2 such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in [a, b]$, and that for any $\epsilon > 0$, $\left|\int_a^b (f_2(x) - f_1(x))dx\right| < \epsilon$.

First, since f_k converges uniformly to f, for any $\epsilon > 0$, there exists K such that if $k > K$, then

$$|f_k(x) - f(x)| < \frac{\epsilon}{4(b-a)}$$

Then since each f_k is integrable, there exists step functions f_{k_1}, f_{k_2} such that

$$f_{k_1}(x) \leq f_k(x) \leq f_{k_2}(x)$$

$$\left|\int_a^b \left(f_{k_2}(x) - f_{k_1}(x)\right) dx\right| < \frac{\epsilon}{2}$$

Define then, $f_1(x) = f_{k_1}(x) - \frac{\epsilon}{4(b-a)}$ and $f_2(x) = f_{k_2}(x) + \frac{\epsilon}{4(b-a)}$.

If $k > K$, then we do have

$$f(x) \leq f(x) + \left(-f_k(x) + f_{k_2}(x)\right) = \left(f(x) - f_k(x)\right) + f_{k_2}(x) < f_2(x)$$

$$f(x) \geq f(x) + \left(-f_k(x) + f_{k_1}(x)\right) = \left(f(x) - f_k(x)\right) + f_{k_1}(x) > f_1(x)$$

Then

$$\int_a^b (f_2(x) - f_1(x))dx = \int_a^b \left(\left(f_{k_2}(x) - f_{k_1}(x)\right) + \left(\frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2(b-a)}\right)\right) dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

As desired. To show that $\int_a^b f_k(x)dx \to \int_a^b f(x)dx$, we must show for $\epsilon > 0$, there exists K such that $k > K$ implies $\left|\int_a^b f_k(x)dx - \int_a^b f(x)dx\right| < \epsilon$. Again, by uniform convergence of f_k, there exists a K_0 such that $|f_k - f| < \frac{\epsilon}{b-a}$ for $k > K_0$. Let $K = K_0$, then

$$\left|\int_a^b f_k(x)dx - \int_a^b f(x)dx\right| = \left|\int_a^b (f_k(x) - f(x))dx\right| < \int_a^b \frac{\epsilon}{b-a}dx = \frac{\epsilon}{b-a}(b-a) = \epsilon$$

Additional Problem 2:

Prove that if $\sum_k g_k$ is a series of functions that converges uniformly on a set S and if h is a bounded function on S, then

$$\sum_k hg_k$$

converges uniformly on S.

Solution:
Since h is bounded on S, there exists an $M > 0$ such that $\sup_{x\in S}|h(x)| = M$. Additionally, since $\sum_k g_k$ is a series of functions that converges uniformly on a set S, we know by the Cauchy-criterion that given any $\epsilon > 0$ there exists N_ϵ such that whenever $n > m > N_\epsilon$ we get that

$$\sup_{x\in S}\left|\sum_{k=m}^{n} g_k(x)\right| < \frac{\epsilon}{M}.$$

We need to show that for any $\epsilon > 0$ there exists an N such that whenever $n > m > N$ we get

$$\sup_{x\in S}\left|\sum_{k=m}^{n} h(x)g_k(x)\right| < \epsilon.$$

But

$$\begin{aligned}
\sup_{x\in S}\left|\sum_{k=m}^{n} h(x)g_k(x)\right| &= \sup_{x\in S}\left|h(x)\sum_{k=m}^{n} g_k(x)\right| \\
&= \sup_{x\in S}\left(|h(x)|\left|\sum_{k=m}^{n} g_k(x)\right|\right) \\
&\le \sup_{x\in S}(|h(x)|)\sup_{x\in S}\left(\left|\sum_{k=m}^{n} g_k(x)\right|\right) \\
&< M\frac{\epsilon}{M} = \epsilon.
\end{aligned}$$

Hence, by letting $N = N_\epsilon$, the proof follows.

Additional Problem 3:

Let $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$ and $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$ for all $x \in \mathbb{R}$.

a. Prove that $s' = c$ and $c' = -s$.

$$s'(x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = \frac{d}{dx}x - \frac{d}{dx}\left(\frac{x^3}{3!}\right) + \frac{d}{dx}\left(\frac{x^5}{5!}\right) - \frac{d}{dx}\left(\frac{x^7}{7!}\right) + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = c(x)$$

$$c'(x) = \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{x^2}{2!}\right) + \frac{d}{dx}\left(\frac{x^4}{4!}\right) - \frac{d}{dx}\left(\frac{x^6}{6!}\right) + \cdots$$

$$= 0 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = -s(x)$$

b. Prove that $(s^2 + c^2)' = 0$.

$$(s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0$$

c. Prove that $s^2 + c^2 = 1$

$$s^2 + c^2 = \int (s^2 + c^2)' dt + (s^2 + c^2)(0) = 0 + 1 = 1$$

Since the integral over 0 is 0, and the only constant in $(s^2 + c^2)$ is 1.