2. For which real numbers $\alpha > 0$ is the function $f : E^2 \to \mathbb{R}$ that is given by $f(x,y) = (x^2+y^2)^\alpha$ differentiable?

Solution:

It suffices to show for which values of α the partial derivatives exist and are continuous on any open interval of E^2. The partial derivatives are given by

$$\frac{\partial f}{\partial x} = 2\alpha x\left(x^2+y^2\right)^{\alpha-1}$$
$$\frac{\partial f}{\partial y} = 2\alpha y\left(x^2+y^2\right)^{\alpha-1}.$$

Whenever $\alpha \geq 1$, both partials are the compositions of continuous functions at all points in E^2 and therefore the function f is differentiable for this case. Whenever $0 < \alpha < 1$ we have that

$$\frac{\partial f}{\partial x} = 2\alpha\frac{x}{\left(x^2+y^2\right)^{1-\alpha}}$$
$$\frac{\partial f}{\partial y} = 2\alpha\frac{y}{\left(x^2+y^2\right)^{1-\alpha}}$$

which is not defined on $(x,y) = (0,0)$. Therefore the function is not differentiable at every point in E^2 for $0 < \alpha < 1$.

3. Show that if f is a real-valued function on a connected open subset of E^n and $f_1' = f_2' = \cdots = f_n' = 0$ then f is constant.

Solution:

Since the partials of exist and are continuous on an open (connected) subset U of E^n by the theorem on page 197 of IA by Rosenlicht, we know f is differentiable. Hence we can apply the lemma on page 196 of IA by Rosenlicht to get that for any $a \in U$

$$\begin{aligned} f(x) - f(a) &= f_1'(x)(x_1-a_1) + f_2'(x)(x_2-a_2) + \cdots + f_n'(x)(x_n-a_n) \\ &= 0\cdot(x_1-a_1) + 0\cdot(x_2-a_2) + \cdots + 0\cdot(x_n-a_n) = 0. \end{aligned}$$

for all $x \in U$. This implies that $f(x) = f(a)$ for any $a \in U$ and for all $x \in U$. Since the derivative exists on all of the open connected subset, we know there are no discontinuities in the function values and the function needs to therefore be constant.

Problem 5:

Proof. Let f be a real-valued function on an open subset U of E^n. Suppose that f is continuously differentiable. Then we have:

$$\lim_{x\to y}\frac{|f(x) - (f(y) + f_1'(y)(x_1-y_1) + \ldots + f_n'(y)(x_n-y_n))|}{d(x,y)} = 0$$

Since $d(x,y) = ((x_1-y_1)^2 + \ldots + (x_n-y_n)^2)^{\frac{1}{2}} \leq |x_1-y_1| + \ldots + |x_n-y_n|$, if we define the function $\epsilon(x,y) = \frac{f(x)-(f(y)+f_1'(y)(x_1-y_1)+\ldots+f_n'(y)(x_n-y_n))}{|x_1-y_1|+\ldots+|x_n-y_n|}$ for $x \neq y$ and $\epsilon(y,y) = 0$, we have

$\lim_{x\to y}\epsilon(x,y)=0$. Thus: $f(x)=f(y)+f_1'(y)(x_1-y_1)+\ldots+f_n'(y)(x_n-y_n)+\epsilon(x,y)(|x_1-y_1|+\ldots+|x_n-y_n|)$ for all $x\in U$. Defining: $A_i(x,y)=f_i'(y)+\epsilon(x,y)$ (or subtraction if $x_i-a_i\le 0$ implies that $f(x)-f(y)=A_1(x,y)(x_1-y_1)+\ldots+A_n(x,y)(x_n-y_n)$ and $\lim_{x\to y}A_i(x,y)=A_i(y,y)=f_i'(y)$; therefore, they are continuous for any $y\in U$.

Now, suppose that there exist continuous real-valued functions $A_i(x,y)$ functions $(i=1,\ldots,n)$ such that $f(x)-f(y)=A_1(x,y)(x_1-y_1)+\ldots+A_n(x,y)(x_n-y_n)$ for all $x,y\in U$. Then for $x,y\in U, x\neq y$, we have $\frac{|f(x)-(f(y)+A_1(x,y)(x_1-y_1)+\ldots+A_n(x,y)(x_n-y_n)|}{d(x,y)}$ $=\frac{|(A_1(x,y)-A_1(y,y))(x_1-y_1)+\ldots+(A_n(x,y)-A_n(y,y))(x_n-y_n)|}{d(x,y)}\le|A_1(x,y)-A_1(y,y)||\frac{|x_1-y_1|}{d(x,y)}+\ldots+|A_n(x,y)-A_n(y,y)||\frac{|x_n-y_n|}{d(x,y)}\le|A_1(x,y)-A_1(y,y)|+\ldots+|A_n(x,y)-A_n(y,y)|$. By the continuity of the $A_i(x,y)$ functions:

$$\lim_{x\to y}|A_1(x,y)-A_1(y,y)|+\ldots+|A_n(x,y)-A_n(y,y)|=0$$

Therfore:

$$\lim_{x\to y}\frac{|f(x)-(f(y)+A_1(x,y)(x_1-y_1)+\ldots+A_n(x,y)(x_n-y_n))|}{d(x,y)}=0$$

Problem 6:

We can apply the change of variable corollary to the fundamental theorem of calculus to get

$$\begin{aligned}F(y)=\int_{\alpha(y)}^{\beta(y)}f(x,y)\,\mathrm{d}x&=\int_{\alpha(y)}^{\alpha(y_0)}f(x,y)\,\mathrm{d}x+\int_{\alpha(y_0)}^{\beta(y_0)}f(x,y)\,\mathrm{d}x+\int_{\beta(y_0)}^{\beta(y)}f(x,y)\,\mathrm{d}x\\&=\int_{\alpha(y_0)}^{\beta(y_0)}f(x,y)\,\mathrm{d}x+\int_{y_0}^{y}f(\beta(x),y)\beta'(x)\,\mathrm{d}x-\int_{y_0}^{y}f(\alpha(x),y)\alpha'(x)\,\mathrm{d}x\end{aligned}$$

for arbitrary $y_0,y\in U$. Evaluating the derivative of F at y_0 yields

$$F'(y_0)=\lim_{y\to y_0}\frac{F(y)-F(y_0)}{y-y_0}=$$

$$\lim_{y\to y_0}\frac{\int_{\alpha(y_0)}^{\beta(y_0)}f(x,y)\,\mathrm{d}x+\int_{y_0}^{y}f(\beta(x),y)\beta'(x)\,\mathrm{d}x-\int_{y_0}^{y}f(\alpha(x),y)\alpha'(x)\,\mathrm{d}x-\int_{\alpha(y_0)}^{\beta(y_0)}f(x,y_0)\,\mathrm{d}x}{y-y_0}$$

$$=\lim_{y\to y_0}\frac{\int_{\alpha(y_0)}^{\beta(y_0)}f(x,y)-f(x,y_0)\,\mathrm{d}x}{y-y_0}+\lim_{y\to y_0}\frac{\int_{y_0}^{y}f(\beta(x),y)\beta'(x)\,\mathrm{d}x-\int_{y_0}^{y}f(\alpha(x),y)\alpha'(x)\,\mathrm{d}x}{y-y_0}$$

For the first term, since f is continuous and is being evaluated on the compact set $[\alpha(y_0),\beta(y_0)]$, we know that it is in fact uniformly continuous and we can therefore swap the limit and integral operations since they act on different variables to get

$$\begin{aligned}&=\int_{\alpha(y_0)}^{\beta(y_0)}\lim_{y\to y_0}\frac{f(x,y)-f(x,y_0)}{y-y_0}\,\mathrm{d}x+\lim_{y\to y_0}\frac{\int_{y_0}^{y}f(\beta(x),y)\beta'(x)\,\mathrm{d}x-\int_{y_0}^{y}f(\alpha(x),y)\alpha'(x)\,\mathrm{d}x}{y-y_0}\\&=\int_{\alpha(y_0)}^{\beta(y_0)}\frac{\partial f}{\partial y}(x,y_0)\,\mathrm{d}x+\lim_{y\to y_0}\frac{\int_{y_0}^{y}(f(\beta(x),y)\beta'(x)-f(\alpha(x),y)\alpha'(x))\,\mathrm{d}x}{y-y_0}\end{aligned}$$

but by the fundamental theorem of calculus the second term becomes

$$=\int_{\alpha(y_0)}^{\beta(y_0)}\frac{\partial f}{\partial y}(x,y_0)\,\mathrm{d}x+f(\beta(y_0),y_0)\beta'(y_0)-f(\alpha(y_0),y_0)\alpha'(y_0).$$

Since $y_0\in U$ was picked arbitrarily, the identity follows.

Problem 7

Let V, W be normed vector spaces, let U be an open subset of V and let $a \in U$. A function $f : U \to W$ is differentiable at a if there exists a continuous linear transformation $T : V \to W$ such that

$$\lim_{x \to a} \frac{||f(x) - f(a) - T(x-a)||}{||x-a||} = 0$$

(a) Let f be differentiable at a and suppose S and T are continuous linear transformations as defined above.

$$||(S-T)(x-a)|| \leq ||f(x) - f(a) - S(x-a)|| - ||f(x) - f(a) - T(x-a)||$$

And we have $\lim_{x \to a} \frac{||(S-T)(x-a)||}{||x-a||} = 0.$

Let $v \neq 0$ and $x = a + v \in U$ such that $\lim_{v \to 0} \frac{||(S-T)(v)||}{||v||} = 0$ So that we have $S(v) = T(v)$ and there is a unique linear transformation.

(b) Let f be differentiable at a and let φ be a function onto W such that $\lim_{x \to a} \frac{||\varphi(x)||}{||x-a||} = 0.$
Let $f(x) = f(a) + Df(a)(x-a) + \varphi(x)$.
$\lim_{x \to a} \varphi(x) = \lim_{x \to a} Df(a)(x-a) = 0$ so $\lim_{x \to a} f(a) + Df(a)(x-a) + \varphi(x) \Rightarrow \lim_{x \to a} f(x) = f(a)$.
Hence f is continuous at a.

(c) Let $W = E^n$ and suppose f is differentiable at a.

$$T(v) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t} = 0$$

Which implies that the directional derivatives of the component functions f_i of f are differentiable at a

Now suppose the component functions of f are differentiable at a. The derivative of the component functions $f_i'(a)$ would be the projection of $f'(a)$ on the i^{th} coordinate of W.

$$|y_i| \leq |y| \leq \sum |y_i| \, \forall (y_1, ..., y_n) \in W$$

Hence f is differentiable at a.

(d) Let V, W, Z be normed vector spaces. U and U' are open subsets of V and W respectively. Let $f : U \to U'$ be a differentiable function at $a \in U$ and $g : U' \to Z$ be a differentiable function at $f(a)$.

Let $f(x) = f(a) + f'(a)(x-a) + \varphi(x)$ and $g(y) = g(f(a)) + g'(f(a))(y - f(a)) + \psi(y)$.
$\lim_{x\to a} \frac{|\varphi(x)|}{|x-a|} = \lim_{y\to a} \frac{|\psi(y)|}{|y-f(a)|} = 0$

$$g(f(x)) = g(f(a)) + g'(f(a))(f'(a)(x-a) + \varphi(x)) + \psi(f(x))$$

$g'(f(a))$ is a linear transformation so there exists a constant C such that for y, $|g'(f(a))(y)| \leq C|y|$. Thus

$$\lim_{x\to a} \frac{|g'(f(a))(\varphi(x))|}{|x-a|} \leq \lim_{x\to a} \frac{C|\varphi(x)|}{|x-a|} = 0$$

Let $\varepsilon > 0$. There exists $\delta > 0$ such that $|\psi(y)| < \varepsilon|y - f(a)|$ whenever $|y - f(a)| < \delta$. f is continuous on a so there exists $\gamma > 0$ such that $|f(x) - f(a)| < \delta$ whenever $|x-a| < \gamma$.

$$|x-a| < \gamma \Rightarrow |\psi(f(x))| \leq \varepsilon|f(x) - f(a)| < \varepsilon|f'(a)(x-a)| + \varepsilon|\varphi(x)|$$

Therefore, there exists a constant C' depending on the linear transformation $f'(a) \frac{|\psi(f(x))|}{|x-a|} \leq \varepsilon C' + \varepsilon \frac{|\varphi(x)|}{|x-a|}$ whenever $|x-a| < \gamma$.

The existence of the above limits imply that $g \circ f$ is differentiable at a and $(g \circ f)' = g'(f(a))f'(a)$.

8. Verify that if $\varphi,\ \psi : \mathbb{R} \to \mathbb{R}$ are twice differentiable functions, if $a \in \mathbb{R}$, and if $f(x,y) = \varphi(x - ay) + \psi(x + ay)$ for all $x, y \in E^2$ then

$$\frac{\partial^2 f}{\partial y^2} = a^2 \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial^2}{\partial y^2}\Big[\phi(x-ay) + \psi(x+ay)\Big] = \frac{\partial}{\partial y}\Big[\frac{\partial}{\partial y}\phi(x-ay) + \frac{\partial}{\partial y}\psi(x+ay)\Big]$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial}{\partial y}\Big[\frac{\partial \phi(x-ay)}{\partial y}(-a) + \frac{\partial \psi(x+ay)}{\partial y}(a)\Big]$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial^2 \phi(x-ay)}{\partial y^2}(-a)^2 + \frac{\partial^2 \psi(x+ay)}{\partial y^2}(a)^2$$

However,

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial^2 \phi(x-ay)}{\partial x^2} + \frac{\partial^2 \psi(x+ay)}{\partial x^2}$$

Therefore,

$$a^2 \frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial^2 f(x,y)}{\partial y^2}$$

9. Verify that the function $u(x,y) = e^{-x^2/4y}/\sqrt{y}$ satisfies that differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{e^{-x^2/4y}(x^2 - 2y)}{4y^{5/2}}$$

$$\frac{\partial u}{\partial x} = \frac{xe^{-x^2/4y}}{2y^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{e^{-x^2/4y}(x^2 - 2y)}{4y^{5/2}}$$

Do the same for the function $\int_a^b f(t)e^{-(x-t)^2/4y}y^{-1/2}dt$ where $[a,b]$ is a closed interval in $\mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is continuous.

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int_a^b f(t)e^{-(x-t)^2/4y}y^{-1/2}dt = \int_a^b \frac{\partial}{\partial y} f(t)e^{-(x-t)^2/4y}y^{-1/2}dt$$

Additional Problem 1:

(a) Prove that if f is bilinear then

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{|(h,k)|} = 0.$$

Let $(e_1^n, ..., e_n^n)$ and $(e_1^m, ..., e_n^m)$ be the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For any $x \in \mathbb{R}^n$, $x = \sum_{i=1}^n x^i e_n^i$ and for any $y \in \mathbb{R}^m$, $y = \sum_{j=1}^m y^j e_m^j$. Thus we have

$$f(x,y) = f(\sum_{i=1}^n x^i e_n^i, \sum_{j=1}^m y^j e_m^j) = \sum_{i=1}^n \sum_{j=1}^m f(x^i e_n^i, y^j e_m^j) = \sum_{i=1}^n \sum_{j=1}^m x^i y^j f(e_n^i, e_m^j)$$

Let $M = \sum_{i,j} ||f(e_n^i, e_m^j)||$ such that

$$||f(x,y)|| = ||\sum_{i,j} x^i y^j f(e_n^i, e_m^j)|| \le \sum_{i,j} |x^i y^j| \, ||f(e_n^i, e_m^j)||$$

$$\le M * \max\{x^i\}\max\{y^j\} \le M||x|| \, ||y||$$

So we have

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{|(h,k)|} \le \lim_{(h,k)\to(0,0)} \frac{M||h|| \, ||k||}{|(h,k)|} = \lim_{(h,k)\to(0,0)} \frac{M||h|| \, ||k||}{\sqrt{||h||^2 + ||k||^2}}$$

Since $||h|| \, ||k|| \le ||h||^2 + ||k||^2$

$$\lim_{(h,k)\to(0,0)} \frac{M||h|| \, ||k||}{\sqrt{||h||^2 + ||k||^2}} \le \lim_{(h,k)\to(0,0)} \frac{M(||h||^2 + ||k||^2)}{\sqrt{||h||^2 + ||k||^2}}$$

$$= \lim_{(h,k)\to(0,0)} M\sqrt{||h||^2 + ||k||^2} = 0$$

(b) Prove that $Df(a,b)(h,k) = f(a,k) + f(h,b)$.

$$\lim_{(h,k)\to(0,0)} \frac{||f(a+h,b+k) - f(a,b) - f(a,k) - f(h,b)||}{||(h,k)||}$$
$$= \lim_{(h,k)\to(0,0)} \frac{||f(a,b) + f(a,k) + f(h,b) + f(h,k) - f(a,b) - f(a,k) - f(h,b)||}{||(h,k)||}$$
$$= \lim_{(h,k)\to(0,0)} \frac{||f(h,k)||}{||(h,k)||} = 0$$

So $Df(a,b)(h,k) = f(a,k) + f(h,b)$.

2. Define $IP : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $IP(x,y) = \langle x, y\rangle$ (the inner product between the vectors x and y).

   (a) Compute $D(IP)(x,y)$ and $(IP)'(x,y)$. We can compute the Jacobian by finding the partials, i.e. using the inner product definition we see that

$$\frac{\partial \langle x,y\rangle}{\partial x} = \frac{\partial x^\top y}{\partial x} = y^\top$$
$$\frac{\partial \langle x,y\rangle}{\partial y} = \frac{\partial x^\top y}{\partial y} = x^\top$$

   thus for column vectors x and y we get that $(IP)'(x,y) = [y^\top, x^\top]$. Hence $D(IP)(x,y)(h,k) = y^\top h + x^\top k$.

   (b) If f and g are differentiable maps from $\mathbb{R} \to \mathbb{R}^n$ and $h(t) = \langle f(t), g(t)\rangle$ show that

$$h'(t) = \langle f'(t)^\top, g(t)\rangle + \langle f(t), g'(t)^\top\rangle.$$

   where $y^\top$ is the transpose of the matrix/vector y.

   Solution:

   The derivative is given by

$$h'(t) = \lim_{k\to 0} \frac{h(t+k) - h(t)}{k} = \lim_{k\to 0} \frac{\langle f(t+k), g(t+k)\rangle - \langle f(t), g(t)\rangle}{k}$$
$$= \lim_{k\to 0} \frac{\langle f(t+k), g(t+k)\rangle - \langle f(t), g(t+k)\rangle + \langle f(t), g(t+k)\rangle - \langle f(t), g(t)\rangle}{k}$$
$$= \lim_{k\to 0} \frac{\langle f(t+k), g(t+k)\rangle - \langle f(t), g(t+k)\rangle}{k} + \lim_{k\to 0} \frac{\langle f(t), g(t+k)\rangle - \langle f(t), g(t)\rangle}{k}.$$

   Breaking this up using the inner product definition yields

$$\lim_{k\to 0} \frac{\langle f(t+k), g(t+k)\rangle - \langle f(t), g(t+k)\rangle}{k} + \lim_{k\to 0} \frac{\langle f(t), g(t+k)\rangle - \langle f(t), g(t)\rangle}{k}$$
$$= \lim_{k\to 0} \frac{f(t+k)^\top g(t+k) - f(t)^\top g(t+k)}{k} + \lim_{k\to 0} \frac{f(t)^\top g(t+k) - f(t)^\top g(t)}{k}$$
$$= \lim_{k\to 0} \frac{(f(t+k) - f(t))^\top g(t+k)}{k} + \lim_{k\to 0} \frac{f(t)^\top (g(t+k) - g(t))}{k}$$
$$= \lim_{k\to 0} \frac{(f(t+k) - f(t))^\top g(t+k)}{k} + \lim_{k\to 0} \frac{(g(t+k) - g(t))^\top f(t)}{k}$$
$$= (f'(t))^\top g(t) + (g'(t))^\top f(t)$$
$$= \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle.$$