11. Prove that if U is an open ball in E^n and $f_1, \dots f_n: U \to \mathbb{R}$ are continuously differentiable functions such that

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

for all $i, j = 1, \dots, n$, then there exists a function $F: U \to \mathbb{R}$ such that $f_i = \partial F/\partial x_i$ for $i = 1, \dots, n$.

As per the hint, let $a = (a_1, \dots, a_n)$ be the center of U, and let F be defined by

$$F(x_1, \dots, x_n) = \int_{a_1}^{x_1} f_1(t, a_2, \dots, a_n)dt + \int_{a_2}^{x_2} f_2(x_1, t, a_3, \dots, a_n)dt + \int_{a_3}^{x_3} f_3(x_1, x_2, t, a_4, \dots, a_n)dt + \cdots$$
$$+ \int_{a_n}^{x_n} f_n(x_1, \dots, x_{n-1}, t)dt$$

Observe

$$\frac{\partial F}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\int_{a_1}^{x_1} f_1(t, a_2, \dots, a_n)dt + \int_{a_2}^{x_2} f_2(x_1, t, a_3, \dots, a_n)dt + \cdots + \int_{a_n}^{x_n} f_n(x_1, \dots, x_{n-1}, t)dt\right)$$
$$= \frac{\partial}{\partial x_i}\int_{a_1}^{x_1} f_1(t, a_2, \dots, a_n)dt + \frac{\partial}{\partial x_i}\int_{a_2}^{x_2} f_2(x_1, t, a_3, \dots, a_n)dt + \cdots + \frac{\partial}{\partial x_i}\int_{a_n}^{x_n} f_n(x_1, \dots, x_{n-1}, t)dt$$

Note that $\frac{\partial}{\partial x_i}\int_{a_j}^{x_j} f_j(x_1, \dots x_{j-1}, t, a_{j+1}, \dots, a_n)dt$ equals 0 if $i > j$ (since the integral is constant with respect to x_i), $f_j(x_1, \dots, x_j, a_{i+1}, \dots, a_n)$ if $i = j$ (by the Fundamental Theorem of Calculus), and if $i < j$, then by the assumptions $\frac{\partial f_i}{\partial x_j}$ exists and is continuous for all pairs i, j, we may pass the derivative inside.

$$= \frac{\partial}{\partial x_i}\int_{a_i}^{x_i} f_i(x_1, \dots, x_{i-1}, t, a_{i+1}, \dots, a_n)dt + \cdots + \int_{a_{i+1}}^{x_{i+1}} \frac{\partial}{\partial x_i} f_{i+1}(x_1, \dots, x_i, t, a_{i+2}, \dots, a_n)dt + \cdots$$
$$+ \int_{a_n}^{x_n} \frac{\partial}{\partial x_i} f_n(x_1, \dots, x_{n-1}, t)dt$$

$$= f_i(x_1, \dots, x_i, a_{i+1}, \dots, a_n) + \int_{a_{i+1}}^{x_{i+1}} \frac{\partial}{\partial x_{i+1}} f_i(x_1, \dots, x_i, t, a_{i+2}, \dots, a_n)dt + \cdots$$
$$+ \int_{a_n}^{x_n} \frac{\partial}{\partial x_n} f_i(x_1, \dots, x_{n-1}, t)dt$$

$$= f_i(x_1, \dots, x_i, a_{i+1}, \dots, a_n) + \big(f_i(x_1, \dots, x_i, x_{i+1}, a_{i+2}, \dots, a_n) - f_i(x_1, \dots, x_i, a_{i+1}, a_{i+2}, \dots, a_n)\big) + \cdots$$
$$+ \big(f_i(x_1, x_2, \dots, x_{n-1}, x_n) - f_i(x_1, \dots, x_{n-1}, a_n)\big) = f_i(x_1, x_2, \dots x_{n-1}, x_n)$$

1. Let $I_1, \ldots, I_N$ be disjoint open intervals in E^n. Show that if $J_1, \ldots, J_M$ are open intervals in E^n such that

$$I_1 \cup \ldots \cup I_N \subset J_1 \cup \ldots \cup J_M$$

Then

$$vol(I_1) + \cdots + vol(I_N) \leq vol(J_1) + \cdots + vol(J_M)$$

Because the intervals I_i are disjoint, we have (by prop. on p288 of text)

$$vol(I_1) + \cdots + vol(I_N) = vol(I_1 \cup \ldots \cup I_N)$$

By the assumption

$$vol(I_1 \cup \ldots \cup I_N) \leq vol(J_1 \cup \ldots \cup J_M)$$

It will suffice now to show $vol(J_1 \cup \ldots \cup J_M) \leq vol(J_1) + \cdots + vol(J_M)$. Proceeding by induction, suppose J_1, J_2 are two, not necessarily distinct sets in E^n. Observe we can write

$$J_1 \cup J_2 = J_1 \cup (J_2 - (J_1 \cap J_2))$$

And

$$J_1 = (J_1 - (J_1 \cap J_2)) \cup (J_1 \cap J_2), J_2 = (J_2 - (J_1 \cap J_2)) \cup (J_1 \cap J_2)$$

The sets above are written as unions of disjoint sets so that we may apply the same proposition. So that $vol(J_1 \cup J_2) = \int_{J_1 \cup J_2} 1 = \int_{J_1} 1 + \int_{J_2 - (J_1 \cup J_2)} 1$ and $vol(J_1) + vol(J_2) = \int_{J_1} 1 + \int_{J_2} 1 = \int_{J_1} 1 + \int_{J_2 - (J_1 \cup J_2)} 1 + \int_{J_1 \cap J_2} 1$. Comparing quantities, we see that $vol(J_1 \cup J_2) \leq vol(J_1) + vol(J_2)$.

Assume that $vol(J_1 \cup \ldots \cup J_{M-1}) \leq vol(J_1) + \cdots + vol(J_{M-1})$. In considering $J_1 \cup \ldots \cup J_{M-1} \cup J_M$, let $S = J_1 \cup \ldots \cup J_{M-1}$, so that $J_1 \cup \ldots \cup J_{M-1} \cup J_M = S \cup J_M$, and apply the base case argument to see that $vol(J_1 \cup \ldots \cup J_M) = vol(S \cup J_M) \leq vol(S) + vol(J_M) = vol(J_1) + \cdots + vol(J_M)$. Then we have

$vol(I_1) + \cdots + vol(I_N) \leq vol(J_1 \cup \ldots \cup J_M) \leq vol(S) + vol(J_M) = vol(J_1) + \cdots + vol(J_M)$, as desired.

7.

a. Let f be a real valued function on a closed interval I of E^n. Show that if f is integrable on I then so is f^2.

f^2 will be integrable if there is a partition P_ϵ such that $U(f^2, P_\epsilon) - L(f^2, P_\epsilon) < \epsilon$. By definition

$$U(f^2, P_\epsilon) - L(f^2, P_\epsilon) = \sum_{A \in P_\epsilon} (\sup(f^2, A) - \inf(f^2, A)) vol(A) \leq \sum_{A \in P_\epsilon} |\sup(f, A)^2 - \inf(f, A)^2| vol(A)$$

$$= \sum_{A \in P_\epsilon} |\sup(f, A) + \inf(\mathrm{f}, \mathrm{A})| |\sup(f, A) - \inf(f, A)| vol(A)$$

By the assumption, since f is integrable, f is bounded, so that $|\sup(f,A) + \inf(f,A)| < M$, for some $M > 0$. Moreover, there exists a partition P such that $U(f,P) - L(f,P) < \frac{\epsilon}{M}$. Let $P_\epsilon = P$. Then

$$U(f^2, P_\epsilon) - L(f^2, P_\epsilon) \leq \sum_{A \in P_\epsilon} |\sup(f,A) + \inf(f,A)||\sup(f,A) - \inf(f,A)| vol(A)$$

$$= M \sum_{A \in P_\epsilon} |\sup(f,A) - \inf(f,A)| vol(A) = M\big(U(f,P) - L(f,P)\big) < M\frac{\epsilon}{M} = \epsilon$$

As desired.

b. Let f, g be real-valued functions on a closed interval I of E^n. Show that if f and g are integrable on I then so is fg.

Observe that we can write $fg = \frac{1}{4}(f+g) - \frac{1}{4}(f-g)$. Further note

- f, g integrable $\Rightarrow f + g$ integrable $\Rightarrow \frac{1}{4}(f+g)$ integrable
- f, g integrable $\Rightarrow f - g$ integrable $\Rightarrow \frac{1}{4}(f-g)$ integrable

Since $\frac{1}{4}(f+g), \frac{1}{4}(f+g)$ integrable, $\frac{1}{4}(f+g) - \frac{1}{4}(f-g)$ will also be integrable

24. Show that if $[a,b]$ is a closed interval in $\mathbb{R}$ and $f : [a,b] \times [a,b] \to \mathbb{R}$ is continuous, then

$$\int_a^b \left(\int_a^x f(x,y)dy \right) dx = \int_a^b \left(\int_y^b f(x,y)dx \right) dy.$$

Proof: By the Fundamental Theorem of Calculus,

Let

$$g(b) = \int_a^b \left(\int_a^x f(x,y)dy \right) dx \text{ then } g'(b) = \int_a^b f(b,y)dy$$

Let

$$h(b) = \int_a^b \big(\int_y^b f(x,y)dx \big) dy, \text{then } h'(b) = \int_a^b f(b,y)dy.$$

Observe that $h(b) = g(b) + c\ \forall b$. Let $b = a$, then we have that

$$\int_a^a \left(\int_a^x f(x,y)dy \right) dx = \int_a^a \left(\int_y^b f(x,y)dx \right) dy + c$$

$$\implies 0 = 0 + c$$

$$\implies c = 0.$$

(we have done this on homework #5)
Therefore, $h(b) = g(b)$ for all b.

1. Let $M_n = \{A : A \text{ is a } n \times n \text{ matrix with entries in } \mathbb{R}\}$. Define

$$f(A) = A^2$$

(a) Compute $Df(A)(H)$;

The derivative is calculated by:

$$\lim_{H \to 0} \frac{||f(A+H) - f(A) - Df(A)(H)||}{||H||}$$

If we let $Df(A)(H) = AH + HA$ we get that:

$$\lim_{H \to 0} \frac{||f(A+H) - f(A) - Df(A)(H)||}{||H||} = \lim_{H \to 0} \frac{||A^2 + AH + HA - A^2 - (AH + HA)||}{||H||}$$

$$= \lim_{H \to 0} \frac{||AH + HA - (AH + HA)||}{||H||} = 0$$

As a sum of linear maps, $AH+HA$ is clearly linear and by the uniqueness of derivatives $Df(A)(H) = AH + HA$.

(b) When $n = 2$ compute $f'(A)$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$f(A) = A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (a^2+bc) & (ab+bd) \\ (ca+cd) & (cb+d^2) \end{bmatrix}$$

The Jacobian matrix of f with respect to A, $f'(A) =$

$$\begin{bmatrix} \frac{\partial f_{11}}{\partial x_{11}} & \frac{\partial f_{11}}{\partial x_{21}} & \frac{\partial f_{21}}{\partial x_{11}} & \frac{\partial f_{21}}{\partial x_{21}} \\ \frac{\partial f_{11}}{\partial x_{12}} & \frac{\partial f_{11}}{\partial x_{22}} & \frac{\partial f_{21}}{\partial x_{12}} & \frac{\partial f_{21}}{\partial x_{22}} \\ \frac{\partial f_{12}}{\partial x_{11}} & \frac{\partial f_{12}}{\partial x_{21}} & \frac{\partial f_{22}}{\partial x_{11}} & \frac{\partial f_{22}}{\partial x_{21}} \\ \frac{\partial f_{12}}{\partial x_{12}} & \frac{\partial f_{12}}{\partial x_{22}} & \frac{\partial f_{22}}{\partial x_{12}} & \frac{\partial f_{22}}{\partial x_{22}} \end{bmatrix} = \begin{bmatrix} 2a & b & c & a+d \\ c & 0 & 0 & c \\ b & 0 & 0 & b \\ a+d & b & c & 2d \end{bmatrix}$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}$. For $x \in \mathbb{R}^n$, the limit

$$\lim_{t \to 0} \frac{f(a+tx) - f(a)}{t}$$

if it exists, is denoted by $D_x f(a)$ and is called the directional derivative of f at a in the direction x.

(a) For e_i an element in the standard basis, show that $\partial_i f = D_{e_i} f$;

Let $e_i = (0, ..., 0, 1, 0, ...0)$, then

$$D_{e_i} f(a) = \lim_{t \to 0} \frac{f(a+te_i) - f(a)}{t} = \lim_{t \to 0} \frac{f(a_1, ..., a_{i-1}, a_i + t, a_{a+1}, ..., a_n) - f(a)}{t} = \partial_i f$$

(b) Show that $D_{sx} f(a) = s D_x f(a)$;

$$D_{sx} f(a) = \lim_{t \to 0} \frac{f(a+tsx) - f(a)}{t} = \lim_{ts \to 0} \frac{(f(a+tsx) - f(a))s}{ts} = s D_x f(a)$$

(c) If f is differentiable at a, show that $D_x f(a) = Df(a)x$.

If f is differentiable at a, then for any non-zero x

$$\lim_{t \to 0} \frac{|f(a+tx) - f(a) - Df(a)(tx)|}{||tx||} = \lim_{t \to 0} \frac{|f(a+tx) - f(a) - t * Df(a)(x)|}{|t|} \frac{1}{||x||}$$

$$= \lim_{t \to 0} \left| \frac{f(a+tx) - f(a)}{t} - Df(a)(x) \right| \frac{1}{||x||} = 0$$

Thus:

$$D_x f(a) = \lim_{t \to 0} \frac{f(a+tx) - f(a)}{t} = Df(a)(x)$$

3. Let $A \subset \mathbb{R}^n$ be an open set and $f : A \subset \mathbb{R}^n \to \mathbb{R}^n$ be a one-to-one continuously differentiable function such that $\det f'(x) \neq 0$ for all $x \in A$. Show that $f(A)$ is an open set and $f^{-1} : f(A) \to A$ is differentiable.

For every $y \in f(A)$, there exists $x \in A$ such that $f(x) = y$ and since f is continuously differentiable and $\det f'(x) \neq 0$, we can apply the Inverse Function Theorem. Let $X \subset A$ be an open set containing x and $Y \subset \mathbb{R}^n$ be an open set containing y such that $Y = f(X)$ which implies $f(A)$ is open.

Then, $f : X \to Y$ has a continuous inverse $f^{-1} : Y \to X$. f is differentiable, so f^{-1} is differentiable at any y, implying that $f^{-1} : f(A) \to A$ is differentiable.

4.

Proof. We know that the derivative of $f : \mathbb{R} \to \mathbb{R}$ is the matrix

$$\begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix}$$

and the determinant of this matrix is

$$\det \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix} = \partial_x f_1 \cdot \partial_y f_2 - \partial_y f_1 \cdot \partial_x f_2$$

Clearly if the derivative is 0, then the determinant is 0 since every element is. To check the other direction, we need to look at the determinant. We know that $\partial_x f_1 = \partial_y f_2$ and $\partial_y f_1 = -\partial_x f_2$ so this becomes

$$(\partial_x f_1)^2 + (\partial_y f_1)^2$$

and since this is the sum of two positive numbers, we know that both must be 0, and therefore, all of the entries of the matrix must be 0. Therefore, $\det f'(x) = 0$ if and only if $f'(x) = 0$ □

5.

Proof. Ler $g(t) = f(tx)$. Then by the theorem on page 198, we know that

$$g'(t) = \sum_{j=1}^{n} \partial_j f(x) \cdot x_j$$

Also, we know that $g(t) = f(tx) = t^m f(x)$ so

$$g'(t) = mt^{m-1} f(x)$$

Combining these two equations we have

$$mt^{m-1} f(x) = \sum_{j=1}^{n} \partial_j f(x) \cdot x_j$$

which, setting $t = 1$ gives

$$\sum_{j=1}^{n} x_j \partial_j f = mf(x) \text{ or } Df(x)x = mf(x)$$

as desired.

□