HW #1 : Solutions

1. a) For $m=7$, $n=37$: $\boxed{q=5 \ \& \ r=2}$ such that:

   $n=qm+r \Leftrightarrow 37=5\cdot 7+2$

   b) For $m=8$, $n=-43$: $\boxed{q=-6 \ \& \ r=5}$ s.t.:

   $-43=-6\cdot 8+5$

   c) For $m=11$, $n=11^{14}+16=11^{14}+11+5=11\left(11^{13}+1\right)+5$:

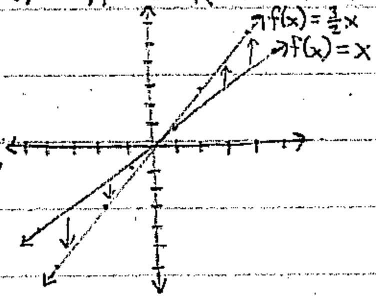
   $\boxed{q=\left(11^{13}+1\right) \ \& \ r=5}$ s.t. $11^{14}+16=\left(11^{13}+1\right)\cdot 11+5$

   d) For $m=4$, $n=20k+9=20k+8+1=4(5k+2)+1$ where $k\in\mathbb{Z}$:

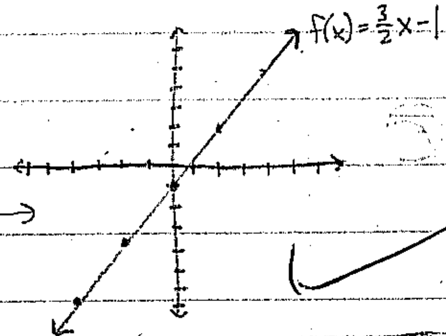
   $\boxed{q=(5k+2) \ \& \ r=1}$ s.t. $n=20k+9=(5k+2)\cdot 4+1=qm+r$

2. a) $f(x)=\frac{3}{2}x-1$:

   Take $f(x)=x$ and stretch it vertically by $\frac{3}{2}$

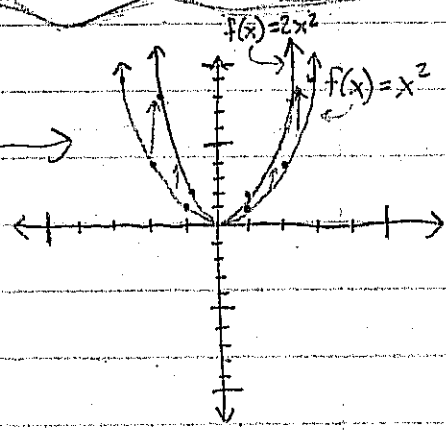


   Now shift it vertically down by 1 →

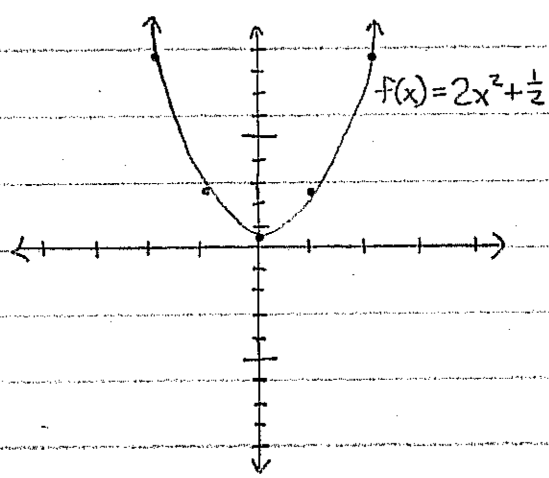


   b) $f(x)=2x^2+\frac{1}{2}$:

   Take $f(x)=x^2$ and stretch it vertically by 2

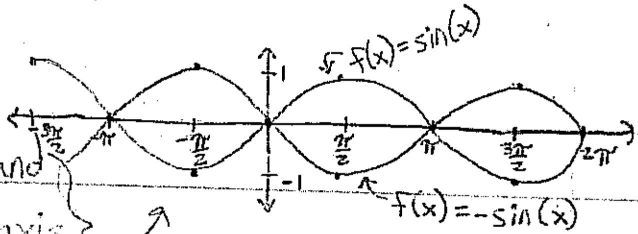


   Now shift it vertically up by $\frac{1}{2}$

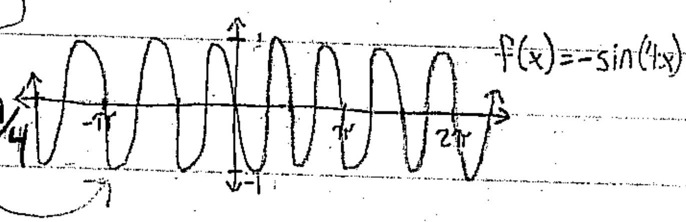


2. c) $f(x) = 1 - \sin(4x)$

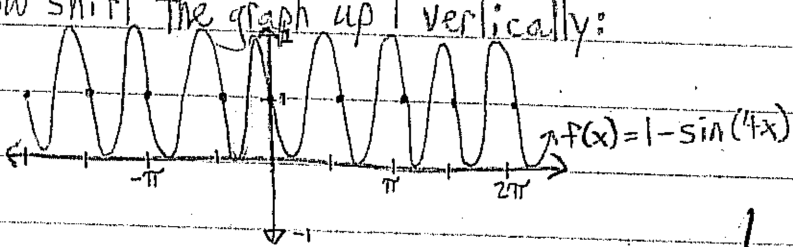
Take $f(x) = \sin(x)$ and flip it over the x-axis (i.e. multiply by -1)



Then compress the graph horizontally by a factor of 1/4



Now shift the graph up 1 vertically:



3 a This is ~~not~~ an equivalence relation.

$a \sim a$: everyone lives in the same county as themself.

$a \sim b \Rightarrow b \sim a$: If a lives in the same county as b, then b lives in the same county as a. They both live in the same county.

$a \sim b,\ b \sim c \Rightarrow a \sim c$: If a lives in the same county as b and b lives in the same county as c, then a lives in the same county as c. All three live in the same county.

An equivalence class is the set of all the people who live in a given county.

b) Set: Q (rational numbers) Relation: $\geq$

reflexive: $x \geq x$ ✓

symmetric $x \geq y$ and $y \geq x$ ✗ No

transitive $x \geq y$, $y \geq z$; $x \geq z$ ✓

No, it is not an equivalence relation. It is not symmetric

c) Set: $\mathbb{Z}$ (integers) Relation: $\equiv$ for n, $k \equiv l$ if $n \mid k-l$

reflexive: $k \equiv k$? $n \mid k-k$ Does n divide zero evenly?

$n \cdot 0 = k-k$, $\frac{0}{n} \in \mathbb{Z}$ yes ✓

symmetric $k \equiv l$ and $l \equiv k$? $n \mid (k-l)$

$n \cdot a = (k-l)$ $a \in \mathbb{Z}$

$-(k-l) = (l-k)$ $n \cdot -a = (l-k)$ $-a \in \mathbb{Z}$

yes ✓

transitive $k \equiv l$, $l \equiv \omega$ does $k \equiv \omega$

$n \cdot a = (k-l)$ $n \cdot b = (l-\omega)$ $k-\omega = (k-l) + (l-\omega)$

$a \in \mathbb{Z}$ $b \in \mathbb{Z}$ $(n \cdot a) + (n \cdot b) = (k-l) + (l-\omega) = (k-\omega)$

$n \cdot (a+b) = (k-\omega)$

$a+b \in \mathbb{Z}$

yes ✓

Yes it is an equivalence relation.
The set of equivalence classes is any pair of integers whose difference has n as one of its factors.

d) Set: Note classes (i.e. C, F, etc) Relation: $N \sim N'$ (if they differ by unison (0 semitones) or a major 3rd (up or down))

reflexive: $N \sim N$ → unison interval. ✓

symmetric: $N \sim N'$, $N' \sim N$ one is a 3rd up, the other is a 3rd down, ✓

transitive: $N \sim N'$, $N' \sim N''$; $N \sim N''$? - Going up by a 3rd twice gets to the same note as going down by a 3rd

lajor
linor

d This is an equivalence relation.

$N \sim N$: because the interval between N and N is the unison interval

$N \sim N' \Rightarrow N' \sim N$: because if the interval from N to N' is unison or a major third up or down, then the interval from N' to N is unison or a major third in the opposite direction.

$N \sim N', N' \sim N'' \Rightarrow N \sim N''$:

if N to N' and N' to N'' are unison, then N to N'' is unison

if N to N' and N' to N'' are major thirds in opposite directions then N to N'' is unison,

if N to N' is unison and N' to N'' is a major third then N to N'' is a major third in the same direction

if N to N' is a major third and N' to N'' is unison then N to N'' is a major third in the same direction

if N to N' and N' to N'' are major thirds in the same direction then N to N'' is a major third in the opposite direction because they would differ by 8 semitones, ignoring octave, which is the same as 4 semitones in the other direction to another note in the same note class as N''.

The equivalence classes are: $\{C, E, A^b\}$, $\{D^b, F, A\}$, $\{D, G^b, B\}$ and $\{E^b, G, B\}$

4 $(a,b) \sim (a,b)$: $ab - ab = 0 \quad \forall (a,b) \in \mathbb{Z}^2, b \neq 0$

$(a,b) \sim (a' b') \Rightarrow (a' b') \sim (a,b)$:

$ab' - a'b = 0 \Rightarrow -ab' + a'b = 0 \Rightarrow a'b - ab' = 0 \Rightarrow (a' b') \sim (a,b)$

$(a,b) \sim (a' b'), (a', b'') \sim (a'', b'') \Rightarrow (a,b) \sim (a'', b'')$

$ab' - a'b = 0, \; a'b'' - a''b' = 0$

$\Rightarrow ab' = a'b, \; a'b'' - a''b' = 0$

$\Rightarrow a' = \frac{ab'}{b}$ as $b \neq 0$

$\Rightarrow \frac{a\cancel{b'}}{b} b'' = a'' \cancel{b'}$

$\Rightarrow ab'' = a''b \Rightarrow ab'' - a''b = 0 \Rightarrow (a,b) \sim (a'', b'')$

3d. (cont) - or go up by a 3rd, then down by a 3rd, making N to N'' a unison interval

Yes, it is transitive ✓

Yes it is an equivalence relation.
The set of equivalence classes includes all the note classes which are a 3rd apart

4. $a, b \in \mathbb{Z}^2$; $(a,b) \sim (a',b')$ if $ab' - a'b = 0$

reflexive : $ab \sim ab$? $ab - ab = 0$ ✓

symmetric : If $ab \sim a'b'$, does $a'b' \sim ab$?

$$ab' - a'b = 0$$
$$a'b - ab' = -(ab' - a'b) = 0$$

transitive : If $ab \sim a'b'$
$a'b' \sim a''b''$
Does $ab \sim a''b''$?

$$ab' - a'b = 0 \quad ab' = a'b \quad \left(\frac{a}{b} = \frac{a'}{b'}\right)$$
$$a'b'' - a''b' = 0 \quad a'b'' = a''b' \quad \left(\frac{a'}{b'} = \frac{a''}{b''}\right)$$
$$\text{Does } ab'' - a''b = 0? \quad ab'' = a''b \quad \left(\frac{a}{b} = \frac{a''}{b''}\right)?$$

Yes, it is an equivalence relation ✓

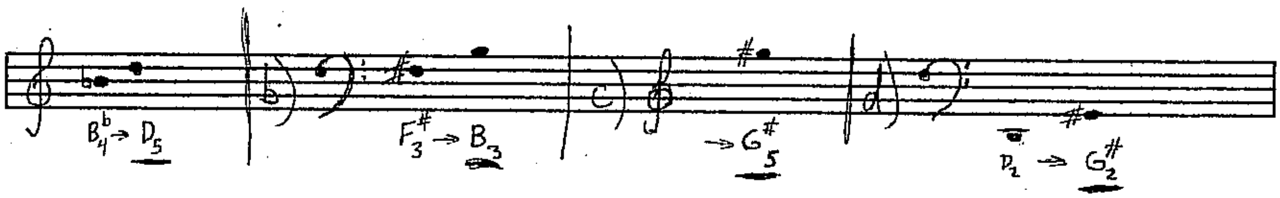
$\mathbb{Q}$ (all rational numbers) is defined as $\frac{a}{b} \mid a, b \in \mathbb{Z} \quad b \neq 0$

and $ab \sim a'b'$ can be written as $\frac{a}{b} = \frac{a'}{b'}$; then any rational number in $\mathbb{Q}$ will have exactly one equivalence class made up of all the equivalent ways to write that number as a fraction (ie. $3/4$ and $6/8$ and $9/12$, etc)

5 a) G_4 b) F_3 c) $D^{\#}_6$ d) B^{b}_1

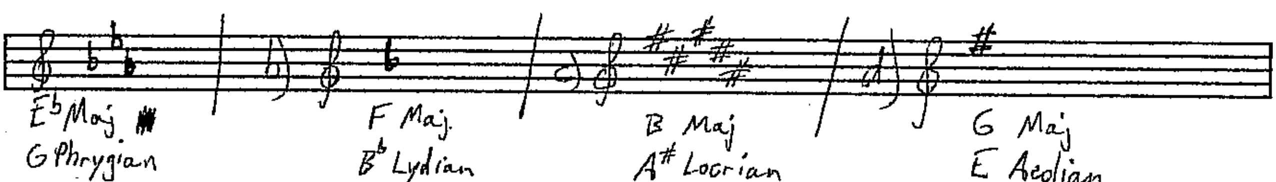
6 a) 5 semitones
A fourth
b) 9 semitones
Maj. 6th
c) 3 semitones
Min. 3rd
d) 11 semitones
Maj. 7th

7. a)



8.

9. a)



10.