BASIC MATHEMATICAL AND MUSICAL CONCEPTS

Sets and Numbers. We assume familiarity with the basic notions of set theory, such as the concepts of *element* of a set, *subset* of a set, *union* and *intersection* of sets, and *function* from one set to another. We assume familiarity with the descriptors *one-to-one* and *onto* for a function.

Following standard convention, we will denote by $\mathbb{R}$ the set of *real numbers,* by $\mathbb{Q}$ the set of rational numbers, and by $\mathbb{Z}$ the set of *integers.* These sets have an ordering, and we will assume familiarity with the symbols $<, \leq, >, \geq$ and basic properties such as: If $a, b, c \in \mathbb{R}$ with $a < b$ and $c > 0$, then $ac < bc$; if $a, b, c \in \mathbb{R}$ with $a < b$ and $c < 0$, then $ac > bc$. We will write $\mathbb{R}^+$ for the set of positive real numbers, $\mathbb{Q}^+$ for the set of positive rational numbers, and $\mathbb{Z}^+$ for the set of positive integers:

$$\begin{aligned}
\mathbb{R}^+ &= \{x \in \mathbb{R} \mid x > 0\} \\
\mathbb{Q}^+ &= \{x \in \mathbb{Q} \mid x > 0\} \\
\mathbb{Z}^+ &= \{x \in \mathbb{Z} \mid x > 0\}.
\end{aligned}$$

The set $\mathbb{Z}^+$ is sometimes called the set of *natural numbers,* also denoted $\mathbb{N}$.

Some Properties of Integers. Given $m, n \in \mathbb{Z}$, we say "m divides n", and write $m|n$, if there exists $q \in \mathbb{Z}$ such that $n = qm$. Grade school arithmetic teaches that for any positive integers m and n, we can divide n by m to get a remainder r having the property $0 \leq r < m$. For example, in the case $m = 9$ and $n = 123$, we have $123 = 13 \cdot 9 + 6$, so $r = 6$. This principle generalizes to the case where n is any integer:

DIVISION ALGORITHM. *Given $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, there exist $q, r \in \mathbb{Z}$ with $0 \leq r < m$ such that $n = qm + r$.*

We will occasionally appeal to one of the axioms of mathematics called the Well-Ordering Principle, which states:

WELL-ORDERING PRINCIPLE. *Any non-empty subset of $\mathbb{Z}^+$ has a smallest element.*

This assertion looks innocent, but cannot be proved without some other similar assumption, so it is taken as an axiom.

Intervals of Real Numbers. We will employ the following standard notation for *intervals* in $\mathbb{R}$: for $a, b \in \mathbb{R}$,

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$
$$[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}.$$

Similarly, we write $(a, b]$ and $[a, b)$ for the half-open intervals.

Functions and graphs. A function from some subset of $\mathbb{R}$ into $\mathbb{R}$ has a *graph,* and we assume familiarity with this notion, as well as the terms *range* and *domain.* We will often use the standard conventions which express a function as $y = f(x)$, where x is the independent variable and y is the dependent variable. When the independent variable parameterizes time, we sometimes denote it by t, so that the function is written $y = f(t)$. A familiar example is $y = mx + b$, where $m, b \in \mathbb{R}$, whose graph is a straight line having slope m and y-intercept b. Another is the function $y = x^2$, whose graph is a parabola with vertex at the origin.
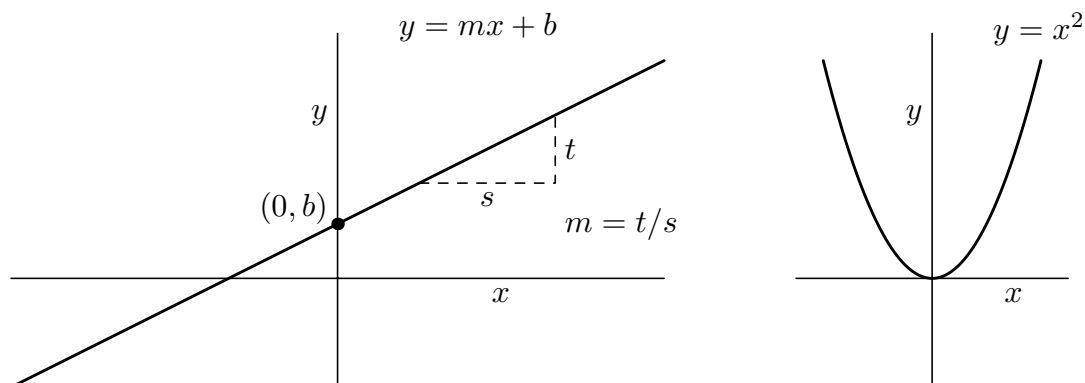
Two functions which will be especially relevant to our topic are the trigonometric functions $y = \sin x$ and $y = \cos x$.
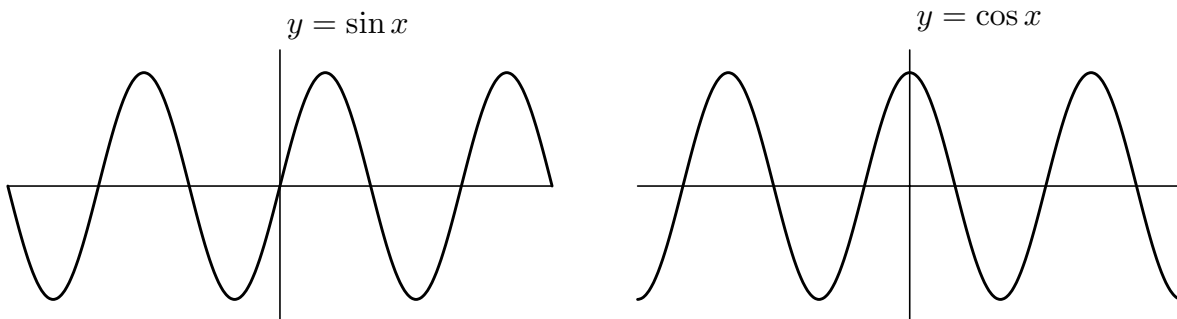

Transformations of Graphs. We will need to understand some procedures which move and deform a graph in certain ways. Let $c \in \mathbb{R}$.

    (1) <u>Vertical shift</u>: The graph of $y = f(x) + c$ is obtained by shifting the graph of $y = f(x)$ upward by a distance of c.

(2) <u>Horizontal shift</u>: The graph $y = f(x - c)$ is obtained by shifting the graph of $y = f(x)$ to the right by a distance of c.

(3) <u>Vertical stretch</u>: The graph of $y = cf(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of a.

(4) <u>Horizontal stretch</u>: The graph of $y = f(x/c)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of a. (Here we assume $c \neq 0$.)

If c in (1) or (2) is a negative number, we must understand that shifting upward (respectively, to the right) by c actually means shifting downward (respectively, to the left) by a distance of $|c| = -c$. If $0 < c < 1$ in (3) or (4) the stretchings are compressions, and if $c < 0$ the stretchings entail a flip about the x-axis in (3), the y-axis in (4).

Below are graphs which illustrate some of these transformations for the function $y = x^2$:

$$y = x^2 \qquad y = x^2 + 1 \qquad y = (x+1)^2$$

$$y = 2x^2 \qquad\qquad y = (x/2)^2$$
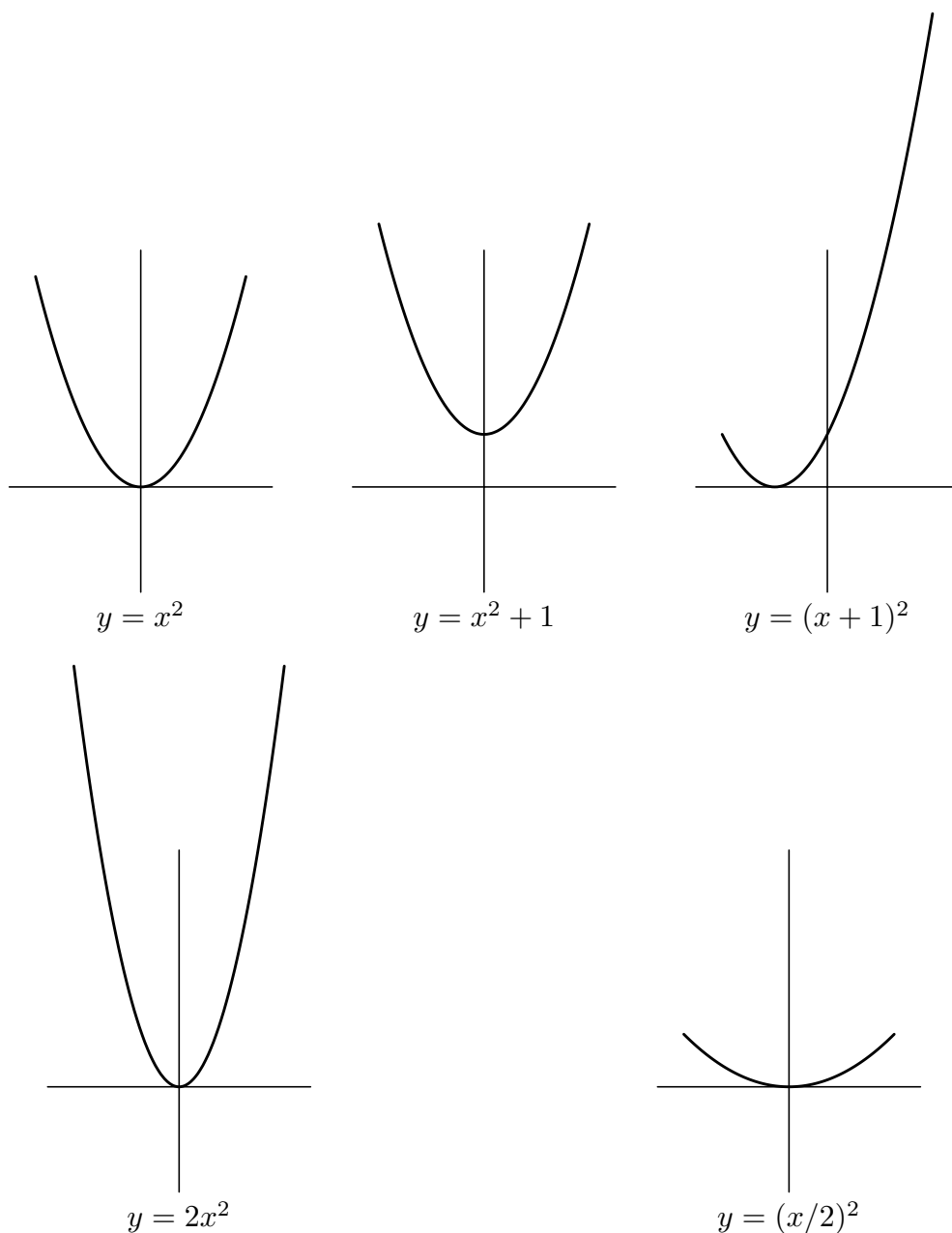
Equivalence relations. Let S be a set an let $\sim$ be a relationship which holds between certain pairs of elements. If the relationship holds between s and t we write $s \sim t$. For example, S could be a set of solid-colored objects and $s \sim t$ could be the relationship "s is the same color as t". We say that $\sim$ is an *equivalence relation* if the following three properties hold for all $s, t, u \in S$:

(1) $s \sim s$ (reflexivity)
(2) If $s \sim t$, then $t \sim s$. (symmetry)
(3) If $s \sim t$ and $t \sim u$, then $s \sim u$. (transitivity)

When these hold, we define the *equivalence class* of $s \in S$ to be the set $\{t \in S \mid t \sim s\}$. The equivalence classes form a *partition* of S, meaning that S is the disjoint union of the equivalence classes.

Pitch. A musical tone is the result of a regular vibration transmitted through the air as a sound wave. The *pitch* of the tone is the frequency of the vibration. Frequency is usually measured in cycles per second, or *hertz,* which is abbreviated *Hz.* For example, standard tuning places the note A above middle C on a musical staff at 440 Hz. It is notated on the treble clef as:

The range of audibility for the human ear is about 20 Hz to 20,000 Hz. We will, however, associate a positive real number x with the frequency x Hz, so that the set of pitches is in one-to-one correspondence with the set $\mathbb{R}^+$.

Notes. In a musical score, specific pitches are called for in a musical score by notes on a staff. We assume familiarity with the usual bass and treble clefs

Middle C as it appears on the treble and bass clefs

and the labeling of notes on the lines and spaces of those clefs using the letters A through G.

We will be needing a concise way to refer to specific notes, hence we will employ the following standard convention: The note C which lies four octaves below middle C is denoted C_0. This note is below the range of the piano keyboard. For any integer n, the C which lies n octaves above C_0 (below C_0 when n is negative) is denoted C_n. Hence middle C is C_4, the C below middle C is C_3, and the lowest C on the piano keyboard is C_1. The other notes will be identified by the integer corresponding to the highest C below that note. Hence the $F^\sharp$ below C_4 is $F_3^\sharp$, while the $F^\sharp$ above C_4 is $F_4^\sharp$. The lowest $B^\flat$ on the piano keyboard is $B_0^\flat$, and the $B^\flat$ in the middle of the treble clef is $B_4^\flat$.

Abstractly, we can envision a keyboard which extends infinitely (and beyond the range of audibility) in both directions, giving an infinite set of notes. This infinite set does not represent all pitches, as there are pitches between adjacent notes. We refer to those notes that appear on the extended keyboard as *keyboard notes.*

Musical Intervals. The *interval* between two notes can be thought of informally as the "distance" between their two associated pitches. (This is to be distinguished from the use of the term 'interval" in mathematics for a subset of $\mathbb{R}$ of the type $(a, b) \subset \mathbb{R}$.) The piano is tuned using *equal temperament* (to be discussed later in detail), which means that the interval between any two adjacent keys (black or white) is the same. This interval is called a *semitone.* The interval of two semitones is a *step,* or *major second,* hence a semitone is a *half-step,* sometimes called a *minor second.* An octave is 12 semitones. Here is a list giving common nomeclature for various intervals:

half-step, or minor second (1 semitone)
step, major second, or whole tone (2 semitones)
minor third (3 semitones)
major third (4 semitones)
fourth, or perfect fourth (5 semitones)
tritone (6 semitones)
fifth, or perfect fifth (7 semitones)
minor sixth, or augmented fifth (8 semitones)
major sixth (9 semitones)
minor seventh, or augmented sixth (10 semitones)
major seventh (11 semitones)
octave (12 semitones)
minor ninth (13 semitones)
ninth (14 semitones)

The meaning of the term "interval" will be made mathematically precise later, but for now we will speak in terms of steps, half steps, fourths, octaves, etc. Also, we will later discuss small modifications of these intervals (e.g., *just* and *Pythagorean* intervals), so to avoid confusion we sometimes refer to the intervals between notes on the abstract infinite keyboard as *keyboard* intervals, or *tempered* intervals. For example we will introduce the Pythagorean major third, which is greater than the keyboard's major third.

We call intervals positive or negative according to whether they are upward or downward, respectively. We sometimes indicate this by using the terms "upward" and "downward" or by using the terms "positive " and "negative" (or "plus" and "minus"). The interval from C_4 to E_3 could be described as down a minor sixth, or as negative a minor sixth.

Octave Equivalence. Music notation and terminology often takes a view which identifies notes that are octaves apart. In this scenario there are only twelve notes on the piano, and "A" refers to any note A, not distinguishing between, say, A_5 and A_1. This is nothing more than a relationship on the set of notes in the chromatic scale: Two notes will be related if the interval between them is an integer multiple of an octave. One easily verifies that the three properties reflexivity, symmetry, and transitivity are satisfied, so that this is in

fact an equivalence relation. We will use the term "modulo octave" in reference to this equivalence relation; hence, for example, $B_2^\flat$ and $B_5^\flat$ are equivalent, modulo octave. A note which is identified by a letter with no subscript can be viewed as an equivalence class by this equivalence relation. Thus $B^\flat$, it can be viewed as the equivalence class of all notes $B_n^\flat$, where $n \in \mathbb{Z}$.

This equivalence of octave identification is similarly applied to intervals: the intervals of a step and a ninth, for example, are equivalent, modulo octave. Each equivalence class of intervals has a unique representative which is positive and strictly less than an octave. (Since intervals are often measured in semitones or steps, this harkens to the mathematical concept modular arithmetic, and later we will make that connection precise.)

In the ensuing discussion of scales and keys we will adopt the perspective of octave indentification.

Scales and key signatures. The standard scale, based on C, is the sequence of ascending notes C D E F G A B C. Since we are using octave equivalence, the last scale note C is redundant; the scale is determined by the sequence C D E F G A B. These are the white keys on the piano keyboard. The whole-step and half-step intervals (modulo octave) between the successive scale notes is:

$$C \xrightarrow{1} D \xrightarrow{1} E \xrightarrow{1/2} F \xrightarrow{1} G \xrightarrow{1} A \xrightarrow{1} B \xrightarrow{1/2} C$$

This sequence $1, 1, \frac{1}{2}, 1, 1, 1, \frac{1}{2}$ of whole-step and half-step intervals initiating with C is incorporated in musical notation, making C the "default" key. One has to be aware of this convention, as there is nothing in the notation itself to indicate that the distance from, say, E to F is a half-step whereas the distance between F and G is a whole step. The above scale can be represented on the treble clef, starting with C_4 (= middle C), as:
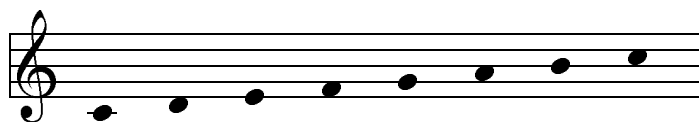
We will say that two sequences of pitches are equivalent if the sequence of respective intervals is the same. Note, for example that the scale contains the two equivalent *tetrachords* (i.e., four note sequences bounded by the interval of a perfect fourth) CDEF and GABC.

We will call any sequence of eight consecutive notes a *standard scale* if it is equivalent to the C scale. Note that the sequence $E^\flat$ F G $A^\flat$ $B^\flat$ C D $E^\flat$ is a standard scale.


One verifies easily that the any ascending sequence of eight consecutive white notes which makes a standard scale must be a C to C sequence. To get a standard scale beginning and ending with a note other than C requires using black notes. The scales F to F and G to G require only one black note. If B is replaced by $B^\flat$, then the F to F scale

$$F \xrightarrow{1} G \xrightarrow{1} A \xrightarrow{1/2} B^\flat \xrightarrow{1} C \xrightarrow{1} D \xrightarrow{1} E \xrightarrow{1/2} F$$

becomes equivalent to the C to C scale, and hence is a standard scale. Similarly if F is replaced by $F^\sharp$, the G to G scale

$$G \xrightarrow{1} A \xrightarrow{1} B \xrightarrow{1/2} C \xrightarrow{1} D \xrightarrow{1} E \xrightarrow{1} F^\sharp \xrightarrow{1/2} G$$

becomes a standard scale. This explains the key signatures for the keys of F and G:
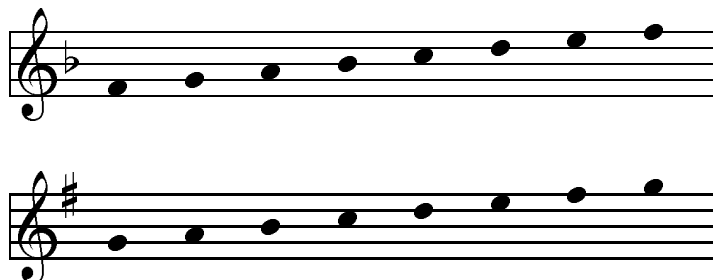
A key signature merely "tailors" notes so as to effect the standard scale in the desired key.

More generally, flatting scale note seven of a standard scale induces a new standard scale based on the fourth note of the original scale. Hence replacing E by $E^\flat$ in the F to F scale yields the $B^\flat$ to $B^\flat$ scale

$$B^\flat \xrightarrow{1} C \xrightarrow{1} D \xrightarrow{1/2} E^\flat \xrightarrow{1} F \xrightarrow{1} G \xrightarrow{1} A \xrightarrow{1/2} B^\flat$$

Hence the key signature of $B^\flat$ is:

Continuing this gives us a sequence of keys C, F, $B^\flat$, $E^\flat$, $A^\flat$, $D^\flat$, $G^\flat$, $C^\flat$. (This sequence continues in theory, but subsequent key signatures will require double flats and eventually other multiple flats.) Note that each successive keynote lies the interval of a fourth (5 semitones) above the previous. Since we are identifying notes an octave apart, it is also correct to say that each successive keynote lies a fifth (7 semitones) below the previous one.

Similarly, sharping scale tone four of a standard scale induces a new standard scale based on the fifth note of the original scale, leading to the sequence of keys C, G, D, A, E, B, $F^\sharp$.

Note that there are two "overlaps" in these two lists of key signatures, in that $C^\flat$ = B and $G^\flat$ = $F^\sharp$.

Diatonic and Chromatic notes. The standard scale is called the *diatonic* scale, whereas the scale containing all the notes is called the *chromatic* scale. Note that the chromatic scale has 12 notes, modulo octave, as opposed to the diatonic scale's 7 notes, modulo octave. In a given key, those notes that lie within the diatonic scale are called *diatonic notes.* They form a subset of the notes of the chromatic scale. (In fact, a scale can be defined as a subsequence of the sequence of chromatic scale notes.)

Cyclic Permutations. Given a finite sequence $x_1, x_2, \ldots, x_n$ of elements in any set, a *cyclic permutation* of this set is obtained by choosing an integer i with $1 \leq i \leq n$, taking

entries $x_1, \ldots, x_1$ from the beginning of the sequence and placing them in order at the end, so as to obtain the sequence

$$x_{i+1}, x_{i+2}, \ldots, x_n, x_1, x_2, \ldots, x_i$$

If we were to arrange the sequence $x_1, x_2, \ldots, x_n$ on a clock with n positions, say, in clockwise fashion with x_1 at the top, then rotate by i positions in the clockwise direction, this cyclic permutation would be obtained by reading off the elements in clockwise fashion, starting from the top. If we chose $i = n$ we would get the original sequence, so that any sequence is a cyclic permutation of itself. The cyclic permutations corresponding to the integers $i = 1, \ldots, n - 1$ are called the *non-trivial cyclic permutations* of $x_1, x_2, \ldots, x_n$.

For example, consider the sequence of numbers $7, 4, 1, 7$. Its cyclic permutations are the sequences $4, 1, 7, 7$, $1, 7, 7, 4$, $7, 7, 4, 1$, and $7, 4, 1, 7$, the first three being the non-trivial ones.

Note that it is possible for a sequence to be a non-trivial cyclic permutation of itself. For example, if we permute the sequence $3, 5, 3, 3, 5, 3$ using $i = 3$, we get the same sequence.

Modality. We have designated the standard scale, in a given key, as a sequence of notes: in C is is the sequence C D E F G A B C. As we pointed out, the last note is redundant, since we are using octave equivalence, so the scale is given by the 7-entry sequence C D E F G A B, and this determines the sequence of adjacent intervals $1, 1, \frac{1}{2}, 1, 1, 1, \frac{1}{2}$ (which still has 7 entries). Consider now a cyclic permutation of the standard scale. For example, consider the sequence E F G A B C D. Note that it also names all the notes which are white keys on the keyboard. It gives the sequence of intervals $\frac{1}{2}, 1, 1, 1, \frac{1}{2}, 1, 1$, which is different from the sequence of intervals for the standard scale. Therefore this sequence beginning with E is not equivalent to the standard scale.

One sees that the sequence of intervals $1, 1, \frac{1}{2}, 1, 1, 1, \frac{1}{2}$ for the standard scale is not equal to any of its non-trivial cyclic permutations, and hence no non-trivial permutation of the standard scale is a standard scale. This underlies the fact that the the seven scales obtained by permuting the standard scale for $i = 1, \ldots, 7$ are distinct.

The term *mode* is used in music to denote the scale in which a musical composition is most naturally accommodated. Quite often the cadences of the piece will arrive at the first scale note, or *tonic* of the composition's mode. Each of the scales derived from the standard scale by a cyclic permutation are modes that were used and named by the Ancient Greeks. These names were incorrectly identified by Glarean in the sixteenth century, yet it was his erroneous ecclesiastical names for the modes which became accepted. They are indicated in the chart below. The left column indicates the scale when played on the keyboard's white notes; the second column is Glarean's name for the scale; the next eight columns name the scale notes when played form C to C.

C-C	Ionian	C	D	E$^\flat$	F	G	A	B	C
D-D	Dorian	C	D	E$^\flat$	F	G	A	B$^\flat$	C
E-E	Phrygian	C	D$^\flat$	E$^\flat$	F	G	A$^\flat$	B$^\flat$	C
F-F	Lydian	C	D	E	F$^\sharp$	G	A	B	C
G-G	Myxolydian	C	D	E	F	G	A	B$^\flat$	C
A-A	Aeolian	C	D	E$^\flat$	F	G	A$^\flat$	B$^\flat$	C
B-B	Locrian	C	D$^\flat$	E$^\flat$	F	G$^\flat$	A$^\flat$	B$^\flat$	C

Musical modes

Note that the key signature determines a unique scale in each of the seven modes. Hence the key signature does not determine the mode: The Ionain key of C has the same signature as the Lydian key of F. To determine the mode of a piece one has to study the music itself.

Major and Minor Modes. By the eighteenth century only two modes were considered satisfactory: the Ionian, which is our standard scale, and the Aeolian. These became known as the *major* and *minor* modes, respectively. With this restriction of possibilities, each key signature represents two possibilities: a major mode and a minor mode. The major mode has as its tonic the first scale note of the standard, or Ionian, scale determined by the key signature and a minor mode has as its tonic the first note of the Aeolean scale determined by the key signature. The minor key which has the same key signature as a given major key is called the *relative minor key* for that major key. The tonic of the relative minor key lies a minor third below that of the corresponding major key's tonic.

For example, no sharps or flats indicates the key of C major and the key of A minor. The character of music determines which mode prevails.

Scale Numbers and Solmization. A basic and traditional use of numerology in music is the numbering of sxale tones. Numbers with a circumflex, or "hat", $\hat{1}$, $\hat{2}$, $\hat{3}$, . . . , will be used to denote specific notes of the diatonic major scale. Thus in the key of A$^\flat$, A$^\flat$ is the first scale tone and is therefore denoted by $\hat{1}$. B$^\flat$ is denoted by by $\hat{2}$, C by $\hat{3}$, and so on.

If we are thinking of the diatonic scale with octave equivalence, only seven numbers are needed. However larger numbers are sometimes used in contexts where octave identification is not being assumed; for example, $\hat{9}$ indicates the diatonic note lying a ninth above some specific scale tonic $\hat{1}$.

Another common practice, called *solmization,* names the scale tones by the syllables *do, re, mi, fa, sol, la, ti.*

do   re   mi   fa   sol   la   ti   do

The chromatic scale tones which lie a half step above or below diatonic notes are denoted by preceding the number with $\sharp$ or $\flat$. Hence, in the key of G, $\flat\hat{6}$ denotes $\text{E}^\flat$. (Note that this can coincide with a diatonic note, e.g., $\sharp\hat{3} = \hat{4}$.) Solmization also provides names for these tones, but we will not give them here.

Exercises

(1) For the following pairs of integers m, n, find the numbers q and r whose existence is asserted in the division algorithm:
   (a) $17, 55$;
   (b) $12, -37$;
   (c) $2, 2^{21} + 3$;
   (d) $7, 14k + 23$, where k is some integer.

(2) Sketch the graphs of these functions:
   (a) $f(x) = \frac{1}{3}x - 1$
   (b) $f(x) = x^2 + 1$
   (c) $f(x) = \cos x + x$

(3) For each of the following sets and relations determine whether or not an equivalence relation has been defined. Explain why or why not.
   (a) The set of people alive now; " has the same mother as".
   (b) $\mathbb{R}$; $\leq$.
   (c) $\mathbb{Z}$; for a fixed positive integer n, $\equiv$ defined by $k \equiv \ell$ iff $n \mid k - \ell$.
   (d) The set of notes on the piano keyboard; $\sim$ defined by $N \sim N'$ iff the interval between N and N' is a major third.

(4) For the set $\{(a, b) \in \mathbb{Z}^2 \mid b \neq 0\}$ show that the relation $\sim$ defined by $(a, b) \sim (a', b')$ iff $ab' - a'b = 0$ is an equivalence relation and that the set of equivalence classes is in one-to-one correspondence with $\mathbb{Q}$.

(5) Identify these notes by letter and subscript (e.g., D_3 or $\text{A}_1^\sharp$):
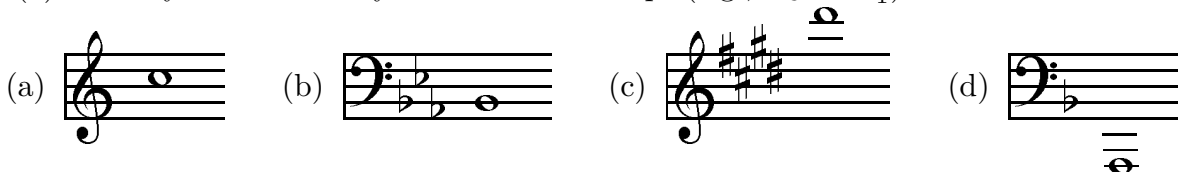
(6) Identify these intervals:

(7) Write on staff paper, and name with subscript, the note which is:
    (a) a minor third above D_2.
    (b) a fifth above $F_2^\sharp$.
    (c) a major ninth below $C_6^\sharp$.
    (d) a tritone below $E_4^\flat$.

(8) Suppose we have an eight-note scale (with first and eighth notes an octave apart) such that the sequence of intervals of adjacent notes contins only the numbers $\frac{1}{2}$ and 1. Is it possible that this scale is a non-trivial standard of itself? If so, give an example. If not, explain why not.

(9) Answer the same question as in the last problem, but for a nine-note scale having the same property.

(10) Notate all the key signatures on a line of staff paper. Order the keys using flats by ascending number of flats and similarly for those using sharps. For each indicate the major key and the minor key it denotes.

(11) For the following modes and tonic notes, indicate the appropriate key signature on staff paper:
    (a) Lydian with tonic G.
    (b) Dorian with tonic $B^\flat$.
    (c) Locrian with tonic $D^\sharp$.
    (d) Phrygian with tonic A.

(12) Transpose this melodic exerpt, written in C minor, up to E minor. Preserve the scale-tone spelling of each melody note.