

## CHAPTER II

### HORIZONTAL STRUCTURE

In mathematics time is often parameterized by a horizontal axis ( $x$ -axis, or  $t$ -axis). Since music is perceived through an interval of time, it is represented visually by its placement along a horizontal axis. On a musical staff the progression from left to right represents the passing of time, while the vertical axis ( $y$ -axis) designates pitch. Thus we refer to its temporal aspects, e.g., the durations of sustained notes and the sequence of events and episodes, as its *horizontal structure*. One way in which most music strives to interest and please the listener is through the presentation of temporal patterns which are satisfying and cohesive. The notation and organization of music's horizontal structure contain a number of relationships with basic mathematical concepts.

**Duration of Notes.** Time durations in music are often measured in *beats*, which are the temporal units by which music is notated. Frequently one beat represents the time interval by which one is prone to “count off” the passing of time while the music is performed, although this is not always the case. The most basic durational unit is, of course, the *note*, and the duration of notes is prescribed by such things as note heads, stem flags, dots, ties, and tuplet designations.

The durational names of notes in Western music is based on the *whole note*, which has a duration in beats (often four) dictated by the *time signature*, which will be discussed later in this chapter. Notes whose duration has proportion  $1/2^n$ ,  $n$  a non-negative integer, with the duration of the whole note are named according to that proportion. Thus, if the whole note has a certain duration in beats, then the *half note* has half that duration, the *quarter note* has one fourth that duration, etc. In the situation where a whole note gets four beats, then a half note gets two beats and the sixty-fourth note represents one sixteenth of a beat.

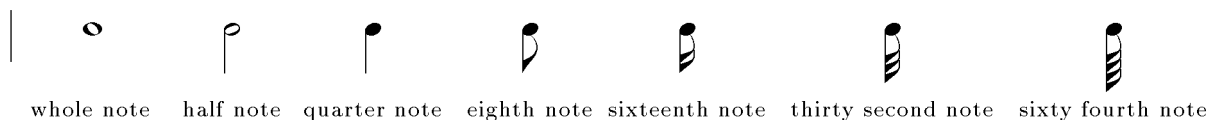
We will use the term *durational note* to mean a note distinguished by its duration, such as half note or quarter note, independent of its associated pitch. Observe that these designations for notes tacitly employ the concept of equivalence class. Here we are declaring two notes to be equivalent if they have the same duration, so that “durational note” refers to the equivalence class of all notes having a given duration (e.g., “half note” designated to the equivalence class of all half notes, regardless of their pitch.) This is to be distinguished from octave equivalence, discussed in Chapter I, whose equivalence classes are called “note classes”.

The pitch of a note is dictated by the vertical position of its notehead on the staff. The duration of the note of the note is dictated by several details which we will discuss

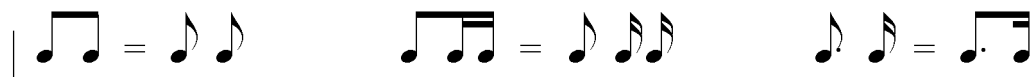
individually. They are:

- (1) whether the interior of the notehead is filled.
- (2) the presence or absence of a note stem, and, if present, the number of flags on the stem or the number of beams attached to the stem.
- (3) the number of dots following the note, if any.
- (4) the tuplet designation of the note, if any.

**Noteheads, Stems, Flags, and Beams.** The whole note and half note are written with an unfilled notehead. For  $n \geq 2$  the  $\frac{1}{2^n}$ -th note is written with a filled notehead. All  $\frac{1}{2^n}$ -th notes except the whole note (i.e., the case  $n = 0$ ) possess a note stem, which either extends upward from the right side of the notehead or downward from the left side of the notehead. For  $n \geq 3$ , the  $\frac{1}{2^n}$ -th note's stem is given  $n - 2$  flags. Thus an eighth note ( $n = 3$ ) has one flag, a sixteenth ( $n = 4$ ) note has two flags, etc.

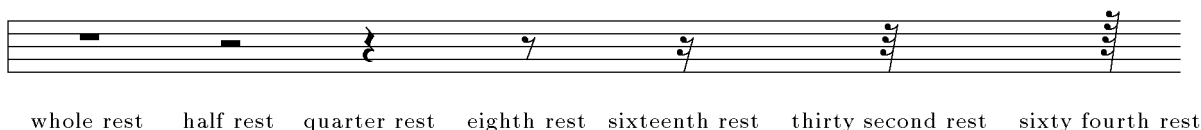


In adjacent notes, flags may be replaced by beams connecting the stems:



(The third example above will be clarified by the section on dots below.)

**Rests.** A *rest* is a notational symbol which indicates silence for a duration determined by the type of rest which appears. Rests come in the same durational types as notes, distinguished by their appearance. The *whole rest*, for example, is a rectangle attached on the underside of a staff line. This and other rests are as indicated below.



The vertical position of rests on a staff is generally as indicated above, but not always. Some circumstances make it more desirable to place them on a higher or lower line, for example when two vocal parts (e.g., soprano and alto) share the same staff and the rest occurs in only one of the parts.

**Dots.** The dot beside a note or rest extends its duration by one half its original duration or equivalently, multiplies the original duration by  $3/2$ . Hence a dotted sixteenth note's duration in beats (still assuming for the moment that the whole note gets four beats) is given by  $\frac{1}{4}(1 + \frac{1}{2}) = \frac{1}{4} \cdot \frac{3}{2} = \frac{3}{8}$ . A second dot beside the note calls for an additional duration of one fourth the original duration (in addition to the extra duration elicited by the first dot), so that, in the above situation, a sixteenth note with two dots has duration

$\frac{1}{4}(1 + \frac{1}{2} + \frac{1}{4}) = \frac{1}{4} \cdot \frac{7}{4} = \frac{7}{16}$ . Although it may seem like a purely academic exercise (since only rarely are more than two dots used), we observe that a note of duration  $d$  followed by  $m$  dots has duration  $d_m$  given by

$$\begin{aligned}
 d_m &= d(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^m}) \\
 &= d \sum_{i=0}^m \left(\frac{1}{2}\right)^i \\
 (1) \quad &= d \left[ \frac{1 - \left(\frac{1}{2}\right)^{m+1}}{1 - \frac{1}{2}} \right] = d \left[ 2 \left( 1 - \left(\frac{1}{2}\right)^{m+1} \right) \right] \\
 &= d \left[ 2 - \left(\frac{1}{2}\right)^m \right] = d \left[ 1 + \frac{2^m - 1}{2^m} \right]
 \end{aligned}$$

The third line in the sequence of equalities above uses the fact

$$(2) \quad \sum_{i=0}^m r^i = 1 + r + r^2 + \cdots + r^m = \frac{1 - r^{m+1}}{1 - r},$$

which holds for any integer  $m \geq 0$  and any real number  $r \neq 1$ . The proof of this will appear as an exercise.

Perhaps the most enlightening expression for  $d_m$  in the above string of equalities is the next to last expression  $d[2 - (\frac{1}{2})^m]$ , which gives:

$$(3) \quad \boxed{\text{A note of duration } d \text{ followed by } m \text{ dots has duration } d_m = d \left[ 2 - \left(\frac{1}{2}\right)^m \right].}$$

It becomes apparent from this formula that the duration of a  $m$ -dotted note approaches  $2d$  as  $m$  becomes large. This is expressed by saying  $[2 - (\frac{1}{2})^m]$  approaches 2 as  $m$  tends to infinity, or

$$\lim_{m \rightarrow \infty} \left[ 2 - \left(\frac{1}{2}\right)^m \right] = 2.$$

It is also apparent that the value of  $d_m$  is always smaller than  $2d$ . The fact that the sums  $\sum_{i=0}^m (\frac{1}{2})^i$  approach 2 as  $m$  gets large is expressed in the equation

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 2.$$

These notions are made precise in calculus.

Let us use the boxed formula (3) above to calculate a certain duration. Suppose we are in a context where a whole note has 2 beats (e.g., when the time signature is  $\frac{2}{2}$ , which will be explained later in this chapter). We then ask: What is the duration of a triply dotted sixteenth note? We first calculate the duration  $d$  of the undotted sixteenth note as

$\frac{1}{16}$ -th of the duration of a whole note, or  $d = \frac{1}{16} \cdot 2 = \frac{1}{8}$ . Here the number of dots is  $m = 3$ , so the formula gives

$$d_3 = \frac{1}{8} \left[ 2 - \left( \frac{1}{2} \right)^3 \right] = \frac{1}{8} \left( 2 - \frac{1}{8} \right) = \frac{1}{8} \cdot \frac{15}{8} = \frac{15}{64}.$$

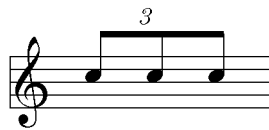
The duration is  $\frac{15}{64}$ -ths of a beat.

**Tuplets.** Note that music’s temporal notation is highly oriented around the prime number 2 and its powers. We do not use terms like “fifth note” or “ninth note.” To divide the  $\frac{1}{2^n}$ -th note into  $k$  equal notes, where  $k$  is not a power of 2, we form a  $k$ -*tuplet* as follows. Find the unique positive integer  $r$  such that

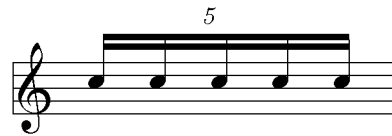
$$2^r < k < 2^{r+1}$$

and notate the tuplet as a group of  $k$   $\frac{1}{2^{n+r}}$ -th notes overset or underset by the integer  $k$ . The resulting tuplet is called the  $\frac{1}{2^{n+r}}$ -*th note*  $k$ -*tuplet*.

For example, suppose we wish to divide the quarter note ( $\frac{1}{2^2}$ -th note) into 3 equal pieces, forming a triplet. Here  $n = 2$ , and since  $2^1 < 3 < 2^2$ , we have  $r = 1$ . We write a sequence of 3  $\frac{1}{2^{2+1}}$ -th notes, or eighth notes, overset by 3, forming an eighth note triple. If, instead, we want to divide the quarter note into 5 notes of equal duration, we note that  $2^2 < 5 < 2^3$ , so  $r = 2$ , so we write a sequence of 5  $\frac{1}{2^{2+2}}$ -th notes, or sixteenth notes, overset by 5. We call this a sixteenth note 5-tuplet.



eighth note triplet



sixteenth note 5-tuplet

The concept of dividing the beat into  $n$  equal parts by an  $n$ -tuplet has an interesting similarity to the notion of the  $n$ th harmonic, which is a vibration  $n$  times faster than the vibration of a fundamental pitch. Harmonics will be discussed in Chapter X.

**Ties and Slurs.** Two notes of the same pitch may be connected by a *tie*, which is a curved line that indicates they are to be considered as one note whose value is the sum of the durations of the two tied notes. Hence, if a whole note gets four beats, then the tying of a quarter note, whose duration is 1, to a dotted sixteenth, whose duration is  $\frac{1}{4}(1 + \frac{1}{2}) = \frac{3}{8}$ ,



gives a duration of  $1 + \frac{3}{8} = \frac{11}{8}$  beats.

Closely related is the *slur*, which looks like a tie but connects notes of different pitches.



This indicates to the performer that he/she should proceed from one pitch to the next with no (or minimal) rearticulation. For example, a violinist interprets this to mean the notes should be played with one stroke of the bow.

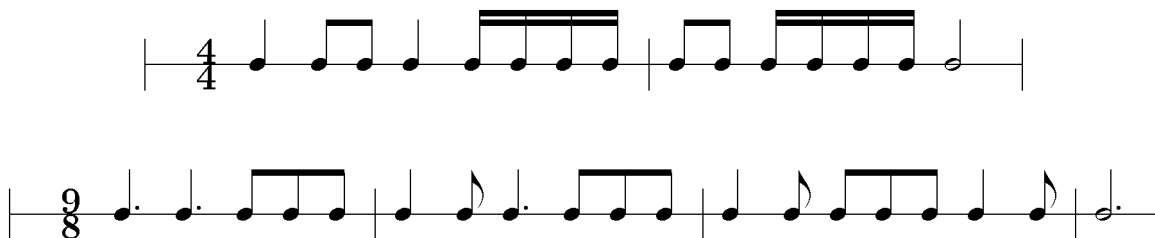
**Meter.** A piece of music is commonly divided into groups of  $n$  beats, for some integer  $n \geq 1$ . Such groups are called *measures* or *bars*. The *meter* of the piece is the number  $n$  of beats per measure together with an assignment of which durational note gets one beat's duration. These parameters are specified by the *time signature* of the piece, which is placed just after the clef symbol (and at subsequent positions if the meter changes). The time signature is comprised of two integers  $\frac{n}{r}$  where  $n \in \mathbb{Z}^+$  and  $r$  is a power of 2. (We refrain from writing  $\frac{n}{r}$  to avoid confusion with fractions.) The meanings of  $n$  and  $r$  are given as follows:

USUAL MEANING. The top number  $n$  specifies the number of beats to a measure and the bottom number  $r = 2^m$  designates that the  $\frac{1}{2^m}$ -th note gets one beat. Thus the time signature  $\frac{2}{4}$  indicated 2 beats to a measure with a quarter note getting one beat.

EXCEPTIONAL CASE. *3 divides  $n$  and  $n > 3$* : In this situation we usually interpret the meter to be a *compound time signature*, which means the number of beats to a measure is taken to be  $n/3$  rather than  $n$ ; thus three  $\frac{1}{2^m}$ -th notes give one beat (where, again,  $r = 2^m$ ). This means that one beat is signified by a dotted  $\frac{1}{2^{m-1}}$ -th note. Thus in  $\frac{6}{8}$  time there are  $6/3 = 2$  beats per measure and one beat is signified by three eighth notes, or a dotted quarter note.

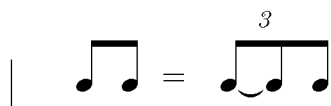
In practice, the integer  $r$  in a time signature  $\frac{n}{r}$  is nearly always 2, 4, or 8.

**Rhythm.** *Rhythm* is the way in which time is organized within measures. Consider these examples:



Upon playing these in tempo (by simply tapping) one observes that a certain amount of musical satisfaction arises from the artistic variation in the ways the measures are filled with durational notes. Rhythms can be straightforward or subtle. Jazz often avoids the obvious by temporarily obscuring the meter using complex sequences.

Sometimes certain types of rhythms are implied but not written, *swing rhythm* being an important example. In a piece where the figure consisting of an eighth note triple with the first two notes tied is pervasive, the triplet notation becomes cumbersome and is often suppressed. The figure is simply denoted by two eighth notes. This is usually indicated by the words “swing rhythm”, or a marking such as



placed above the first measure of the piece. Of course, this rhythm can be notated precisely by using one of the compound time signatures  $\frac{3n}{8}$  and writing it as a quarternote followed by an eighth note.

**Melody.** Melody is the succession of notes (single pitches with prescribed duration) which are most prominent in a musical composition and which serve to define and characterize the piece. Melody is the sequence of notes in a popular song that a solo vocalist sings, while other notes are being played in accompaniment. In a symphony the melody is often (but not always) played by the highest instrument, typically the first violin section.

It should be emphasized that a melody is defined and made recognizable not only by its sequence of pitches, but by its rhythm. This is exemplified by the descending scale in the Ionian (major) mode,



which by itself evokes no particular song. However, the same sequence of pitches set to the rhythm



is immediately recognized as *Joy To The World*.

**Repeating Patterns.** One way music achieves cohesion is through the repetition of certain melodic patterns, often with variation and embellishment. This can mean repeating a major section of the piece or, more “locally”, by juxtaposing brief melodic figures. The former phenomenon will be discussed later under *form*. For the moment, however, we will discuss the local types of repetitions which correspond to the mathematical concept of geometric transformation.

**Translations.** A simple example of such is a horizontal shift, or *translation*, which is effected in the graph of a function  $y = f(x)$  when we replace it by  $y = f(x - c)$  (see Chapter I). This often appears in music as the repetitions (horizontal translation) of the sequence of pitches or the rhythmic pattern. Here is a familiar example which illustrates rhythmic translation:



Get out the way, old Dan Tuck-er! You're too late to come to sup-per.

Note that the rhythm of the first two bars is repeated twice, while the sequence of pitches varies.

An example of melodic (as well as rhythmic) translation is found in the spiritual *When The Saints Go Marching In*,

Oh, when the Saints go march-ing in, Oh, when the Saints

where the melodic sequence F-A-B<sup>b</sup>-C appears three times consecutively. (This example is given in [1].)

**Transposition.** When a repeating pattern is being represented melodically, it is possible to also apply a vertical shift or *transposition*, analogous to replacing the graph of  $y = f(x)$  by that of  $y = f(x) + c$ . Such a shift may repeat a melodic excerpt, transposing each note upward or downward by a fixed interval, as in the first sixteen bars of George and Ira Gershwin’s “Strike Up The Band”, in which the second eight measures repeat the melody of the first, transposed up by the interval of a fourth.

Let the drums roll out! Let the trum- pet call! While the  
 peo- ple shout! Strike up the band! Hear the cym- bals ring! Call- ing  
 one and all! To the mar- tial swing Strike up the band!

Another form of transposition occurs when a diatonic melody is moved up or down by the same number of diatonic scale tones, producing a melody having the same general shape shape, but with intervals not perfectly preserved due to the differing intervals between adjacent diatonic notes. This occurs on the German Carol *O Tannenbaum (O Christmas Tree)*. Note that the the first bracketed sequence below is shifted downward by one diatonic scale tone in the second bracketed sequence.

O Christ-mas tree, O Christ-mas tree, Your col- or is un- chang- ing

When from all trees the col- ors go, You still are green a- midst the snow.

**Retrogression.** Yet another form of transformation in music is *retrogression*, which is analogous to the mathematical notion of horizontal reflection. Such a reflection is exemplified when we replace the graph of  $y = f(x)$  with that of  $y = -f(x)$ , reflecting the graph around the  $y$ -axis. In music, “retrogression” means “inverting the order of notes”, so that the resulting sequence forms a reflection of the initial one. In this excerpt from *Raindrops Keep Falling On My Head* (another example from [1]), note the symmetry of the melody around the centerpoint designated by  $\wedge$ :

Rain-drops keep fall- ing on my head, they

$\wedge$

**Form.** The sequence of larger sections of music into which music may be organized is sometimes called *form*. The number of measures in a section is often a power of 2. For example, ragtime compositions typically consist of three or four sections, each section having 16 measures; sometimes one or more of these sections is repeated once. These sections are distinguishable by the listener by virtue of different rhythmic and melodic character. If a composition consisted of three sections, we might denote the form by: A B C. If the first two sections were repeated, the form would be AA BB C. Scott Joplin’s *Maple Leaf Rag* has the form AA BB A CC DD.

Two classical type forms are *binary form* and *ternary form*. The former presents a piece of music as two main sections which are repeated, giving a form AABB. Many of the minuets and scherzos of the late 18th and 19th centuries have this form. Ternary form presents three sections, with the first and third being the same, or very similar, giving a pattern ABA. It often is found in the nocturnes of Chopin and the piano pieces of Brahms.

Most songs in American popular music and folk music can be naturally divided into 8-bar segments, some which typically recur. One common pattern is AABA, exemplified by the Tin Pan Alley song *Five Foot Two, Eyes of Blue*. If a section bears strong resemblance to another, it may be given the same letter followed by ‘. For example, the form of the song *Edelweiss* is represented by AA’BA’.

**Symmetry.** The word *symmetry* is used in music in general reference to the phenomena of transformations and repeating sections. A compositional goal in many styles of music is to create balanced portions of unity and contrast - enough repetition to give a piece interest and cohesion, but not so much as to make it repetitive or boring. As an example



of a simple piece which illustrates the use multiple symmetries, let us refer back the the carol *O Tannenbaum*. Here the form of the complete chorus is AABA, but each section has internal symmetries as well. Both the A and B sections incorporate melodic and rhythmic transposition, with the B section featuring the downward diatonic transposition discussed earlier in this chapter.

### References

1. Trudi Hammel Garland and Charity Vaughn Kahn, *Math and Music; Harmonious connections*, Dale Seymour Publications, 1995.

### Exercises

- (1) In  $\frac{4}{4}$  time, give the duration in beats for:
  - (a) a dotted thirty-second note
  - (b) a half note with four dots
  - (c) a quarter note tied to a sixteenth note with three dots
- (2) In  $\frac{12}{8}$  time, taken as a compound time signature, give the duration in beats for:
  - (a) a dotted eighth note
  - (b) a quarter note tied to a sixteenth note
  - (c) a thirty-second note with three dots
- (3) Prove the equation:

$$1 + r + r^2 + \cdots + r^m = \frac{1 - r^{m+1}}{1 - r} .$$

for any integer  $m \geq 0$  and any real number  $r \neq 1$ . (Hint: Consider the product  $(1 - r)(1 + r + r^2 + \cdots + r^m)$ .)

- (4) Notate and name the following tuplets:
  - (a) that which divides the quarter note into 5 equal notes
  - (b) that which divides the eighth note into 3 equal notes
  - (c) that which divides the whole note into 11 equal notes
- (5) Notate and give the total duration of:
  - (a) a sixteenth note septuplet
  - (b) a half note triplet
  - (c) a quarter note 17-tuplet
- (6) Complete these measures with a single durational note:

(a)  (b)  (c) 

- (7) Complete the following excerpt three ways with a measure having the same rhythm,



employing, respectively:

- (a) diatonic transposition up one scale tone
- (b) diatonic transposition up three scale tones
- (c) chromatic transposition up a minor third

Which of these, if any, represent both diatonic and chromatic transposition?

- (8) Give the form (e.g., ABAC or ABA) of the following songs (one chorus only):
- (a) *Let Me Call You Sweetheart*
  - (b) *My Bonnie Lies Over The Ocean*
  - (c) *Let It Be*
  - (d) *The Rose*