CHAPTER III

HARMONY AND RELATED NUMEROLOGY

Harmony. *Harmony* is that aspect of music in which different pitches are sounded simultaneously. The earliest harmony in Western music consisted of parallel octaves, fourths and fifths. Over the centuries a rich array of harmonic patterns and clichès has evolved, and later we will examine some of these patterns and the role mathematics played in their development. The basic harmonic building block is the *chord,* which is a collection of notes, usually three or more, sounded simultaneously. Chords have a type which is determined by the intervals, modulo octave, between the notes in the chord. A chord also has a numerical label which is determined by its juxtaposition with the tonic note of the key.

Intervals and Modular Arithmetic. Before launching our discussion of harmony, we introduce the notion of modular integers, which allows us to refine the notion of modular interval as defined in Chapter I.

It was given in an excercise in Chapter I to show that, for a fixed integer $n \in \mathbb{Z}^+$, the relation $k \equiv \ell$ defined by $n \mid (k - \ell)$ is an equivalence relation on the set of integers $\mathbb{Z}$. We express this relationship by saying "k is congruent to ℓ modulo n", or $k \equiv \ell \mod n$. It is easily seen that $k \equiv \ell \mod n$ if and only if k and ℓ have the same remainder r obtained from n using the the Division Algorithm: $k = qn + r$ (see Chapter I), and that each equivalence class containes precisely one of the integers $\{0, 1, 2, \dots, n-1\}$. Hence there are m equivalence classes. We denote the set of equivalence classes by $\mathbb{Z}_n$.

The case $n = 12$ has a special significance in music, as follows. By measuring intervals in semitiones, the set of intervals is identified with the set $\mathbb{Z}$ of integers, with an integer k corresponding to the interval of k semitiones, upward if k is positive, downward if k is negative. With this identification, equivalence modulo 12 is nothing more than octave identification: Two intervals k semitones and ℓ semitones are equivalent modulo octave if and only if $k \equiv \ell \mod 12$. Accordingly, each equivalence class of intervals contains a unique interval of r semitones with $0 \leq r < 12$ (i.e., a non-negative interval less than an octave), and this r is obtained as the remainder in the Division Algorithm with $n = 12$.

For example, the interval of a ninth, which is 14 semitones, is equivalent to the interval of a step, 2 semitones, since $14 \equiv 2 \mod 12$. Similarly, one verifies that down a fourth is equivalent to up a fifth, since $-5 \equiv 7 \mod 12$.

In some contexts when we speak of musical intervals, we actually mean interval classes modulo octave, of which there are twelve. We will try to make this distinction clear at all times. Note that there is a well-defined interval class between any ordered pair of note

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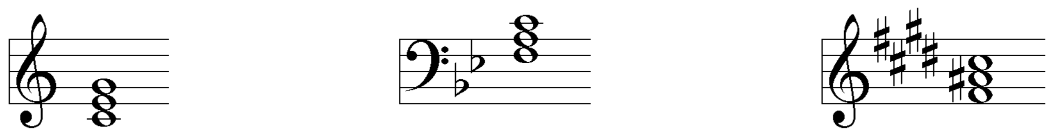
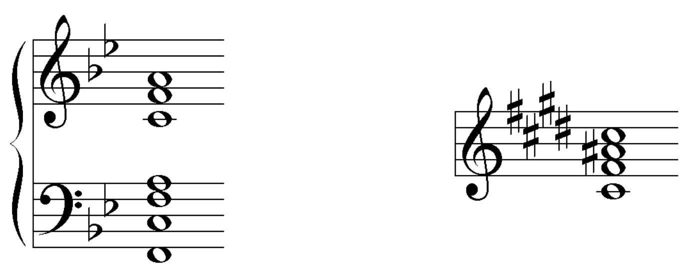
classes, which can be represented uniquely by a non-negative interval less than an octave. For example, the interval from $E^\flat$ to B is represented by 8 semitones, or a minor sixth.

Major Chord. The first chord we will consider is is the *major* chord, which consists of a note sounded simultaneously with the notes which lie a major third and a fifth above the given note. Below are some examples of major chords:



The note of the major chord which has chord notes lying a major third and a fifth above it is called the *root.* The two subsequent notes are called the *third* and *fifth,* respectively. Thus in the middle example above, the root of the major chord is F, the third is A, and the fifth is C.

In general, chords are defined by the note classes (and modular intervals) they employ. Thus any of the notes in a chord may be displaced and/or doubled by the interval of one or more octaves. Hence the following variations are also major chords:



Voicing. The term *voicing* is used to denote the particular way a chord is written, i.e., the specific notes, as opposed to note classes, which are chosen. Observe that the root need not be the bottom note. In the rightmost voicing above, the lowest note is the fifth of the major chord. But observe that, regardless of the voicing, there is no abiguity about which note is the root, third, or fifth of a major chord. That is because, like the standard scale, the sequence of modular intervals $(4, 3, 5)$ (measured in semitones by a elements of $\mathbb{Z}_{12}$) between successive note classes comprising the major chord

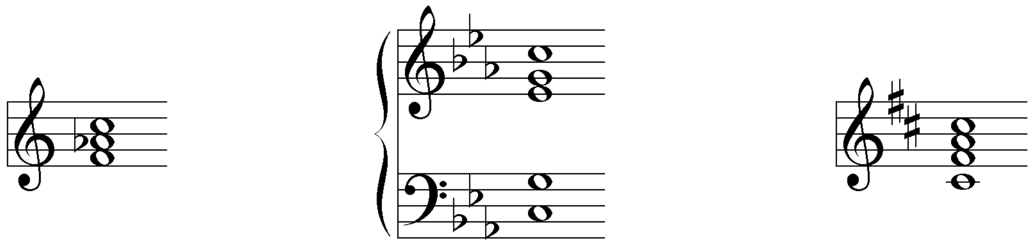
$$\text{root} \xrightarrow{4} \text{third} \xrightarrow{3} \text{fifth} \xrightarrow{5} (\text{root})$$

has the property that no non-trivial cyclic permutation of the sequence gives the same sequence, in other words it has no *non-trivial cyclic symmetries.*

Minor Chord. The *minor* chord is defined by the sequence of modular intervals $(3, 4, 5)$. Thus it consists of a root together with the the notes which lies a minor third and a fifth, modulo octave, above the root. As with the major chord, the two successive tones are called third and fifth,

$$\text{root} \xrightarrow{3} \text{third} \xrightarrow{4} \text{fifth} \xrightarrow{5} (\text{root})$$

and again the root, third and fifth are uniquely determined by the sequence of intervals. Here are some minor chords.



Triads. Chords which contain exactly three notes, modulo octave, are called *triads.* The major and minor chords are examples. Triads have been a fundamental part of harmony in Western music since the seventeenth century. The term *triadic* is sometimes applied to music that primarily features triads.

Diminished and Augmented Chords. Two other triads which play significant roles in Western music are the *diminished* chord, defined by the sequence of intervals $(3, 3, 6)$, and the *augmented* chord, defined by the sequence $(4, 4, 4)$.

diminished augmented

Note that the augmented chord, unlike all the previously introduced chords, has no discernable root. Any cyclic permutation of its sequence of intervals gives the same sequence.

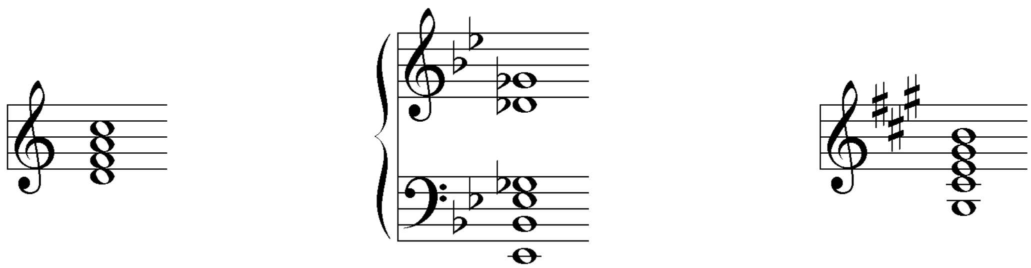
Seventh and Minor Seventh Chords. Finally we introduce two important four-note chords. The first is the *seventh* chord, defined by the sequence of intervals $(4, 3, 3, 2)$. The notes are called root, third, fifth and seventh, respectively.

$$\text{root} \xrightarrow{4} \text{third} \xrightarrow{3} \text{fifth} \xrightarrow{3} \text{seventh} \xrightarrow{2} (\text{root})$$

This sequence has no non-trivial cyclic symmetries, hence the root, third, fifth, and seventh are distinguishable. Examples are:

Observe that this chord contains the major chord with the same root, third, and fifth. We will later say quite a bit about this chord's role in the development of Western harmony and the tuning obstacles associated with it.

The other four-note chord to be introduced here is the *minor seventh,* defined by the sequence $(3,4,3,2)$, which admits no non-trivial cyclic symmetries. Here are examples.



The minor seventh chord contains the minor chord having the same root, third, and fifth.

To avoid any possible confusion between the major chord and the seventh chord (and other chords that contain the major chord), we often refer to the major chord as the *major triad.* Similarly we use the term *minor triad* for the minor chord.

Diminished Seventh. The *diminished seventh,* or *full diminished* chord is defined by the sequence of modular intervals $(3,3,3,3)$. Like the augmented chord, every permutation of its sequence gives the same sequence, so it too has no discernable root. Here are two examples:

diminished

augmented

This chord imparts a feeling of tension or instability. It often resolves to a more consonant chord, such as a major or minor triad.

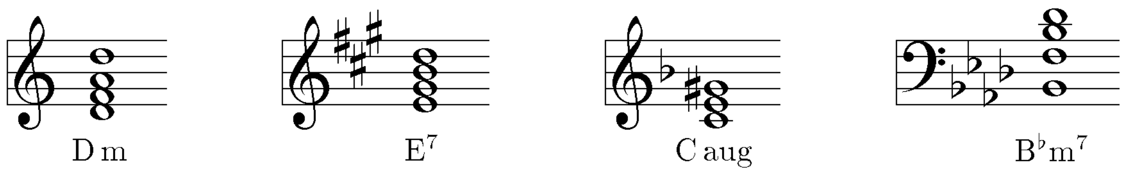
Chord Labeling. Chords are often labeled and/or denoted by identfying the root followed by a suffix which indicates the type of chord. The root can be labeled by identifying a specific note class, such as D or B$^\flat$, or a scale tone. In the later case the scale tone is indicated by Roman numeral, possibly preceded by $\sharp$ or $\flat$, such as III or $\flat$VI. For this the proper mode must be incorporated.

There are various conventions for writing the suffix which indicates the chord type. We will use the following notations for the suffixes:

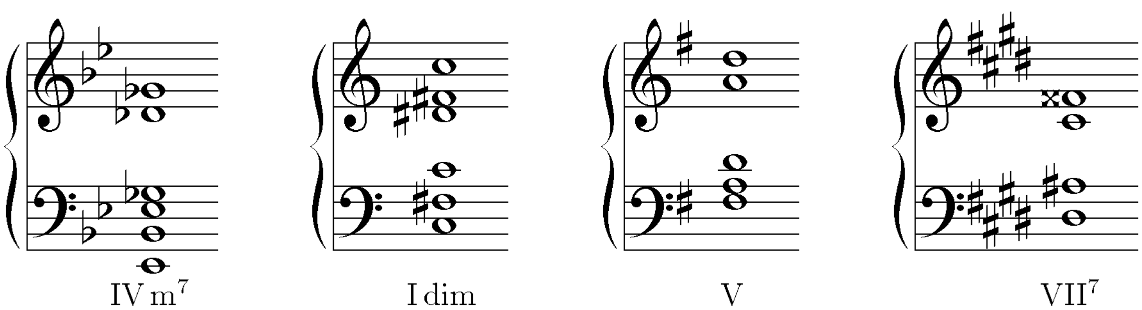
major triad: no suffix
minor triad: m
augmented: aug
diminished: dim
seventh: ^{7}
minor seventh: m^7
diminished seventh: ^{07}

For example, a major triad whose root is C is denoted by C. In the key of F major, it would be denoted V. In the key of A minor, it would be denoted III. The minor seventh chord whose root is F♯ is denoted F♯m^7. In the key of D major, it would be IIIm^7. In the key of G minor, it would be denoted ♯VII. We often label augmented or diminished seventh chords, which have no discernable root, by declaring the root to be the lowest note in its voicing.

Here are some chords labeled according to the note class of the root.



Below are some chords labeled according to the scale tone numeral of the root. Here we assume the major (Ionian) mode.



Later we will give mathematical reasons why certain chords seem to possess a "harmonious", or consonant, quality, while others have a more "clashing", or dissonant, effect.

Progressions. Musical "character" is created by the way various chord types are organized and juxtaposed in real time. The procedure from one chord to the next is called *progression.* A certain amount of musical satisfaction is obtained merely from a pleasing or catchy sequence of progressions. Certain patterns are common, thus giving musical clichés are quite familiar to most listeners. An example is a progression in which the root moves counter-clockwise around the circle of fifths, i.e., the root of a chord is a fourth above the root of the preceding chord, as in the sequence of major mode progressions:

$$\mathrm{VI}^7 \longrightarrow \mathrm{II\,m} \longrightarrow \mathrm{V}^7 \longrightarrow \mathrm{I}$$

Often a melody suggests the chords which should underlie it by offering a sequence of notes which lie mostly within a certain chord. Sometimes this basic "implied harmony" can be artistically altered or enhanced. For example, this melody, in F major,

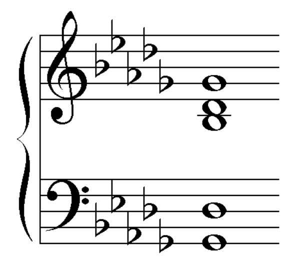
is comfortably accommodated by the sequence I $\to$ V^7 $\to$ I (or F $\to$ C^7 $\to$ F), each chord sustained or played in arpeggio for one measure. All the melody notes lie within the respective chords except for the D in the second measure, which doesn't lie in C^7. But observe that the following harmonizations also work,

I $\longrightarrow$ II m $\longrightarrow$ III m
I $\longrightarrow$ IV $\longrightarrow$ V
I $\longrightarrow$ ♭VII $\longrightarrow$ I
I $\longrightarrow$ ♭VII $\longrightarrow$ V

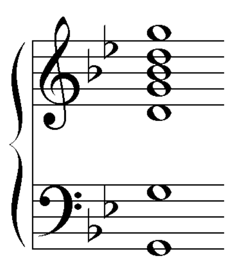
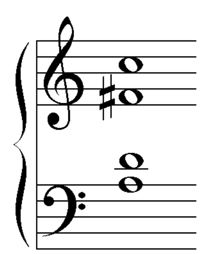
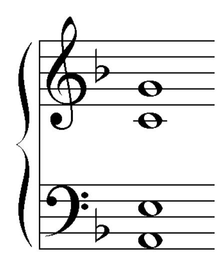
each giving the passage a different personality.

Exercises

(1) Identify these chords by root note and suffix (e.g., B m^7 or $E^\flat$ aug):

(a)  (b)  (c)  (d) 

(2) Identify these chords by root scale note and suffix (e.g., V^7 or ♯II m). Assume the major mode in (a) and (b), the minor mode in (c), and the Dorian mode in (d).

(a)  (b)  (c)  (d) 

(3) Write these chords with correct spelling on the treble clef:

(a) Dm^7 (b) $E^\flat$dim (c) $A^\flat$ (d) $F^{\sharp 7}$

(4) Write these chords with correct spelling on the bass clef, using the indicated (major) key signature:
(a) II^7 in the key of D major
(b) IVm^7 in the key of $A^\flat$major
(c) I aug in the key of F major
(d) ♭VII in the Lydian key of $E^\flat$

(5) For each of the types of chords discussed in the text, list by Roman numeral all the ways the chord can be created using only <u>diatonic</u> note classes in the major mode.

(6) Consider the chord obtained by taking the seventh chord and flatting its fifth. Such a chord is sometimes labeled with the suffix $^{7-5}$. Show that this chord does not have a discernable root. Write an an example of such a chord and give all possible labelings of it.