RATIOS AND MUSICAL INTERVALS

We like to think of an interval as the "distance" between two pitches. The most basic interval is the octave. If one hears the pitches 440 Hz (A_4) and 880 Hz, one recognizes the latter as being one octave above the former, hence 880 Hz is A_5. The pitch 220 is one octave below A_4, hence is A_3. The difference between the frequencies of A_3 and A_4 is 220, while the difference between the frequencies of A_4 and A_5 is 440, yet the intervals are the same – one octave. This reflects the fact that the octave corresponds to a factor of 2, and that an interval should not be associated with the difference between the two frequencies, but rather the <u>ratio</u> between the two frequencies.

The Equivalence Relation of Ratios. Consider the relation on the set of ordered pairs from $\mathbb{R}^+$ (i.e., the set $(R^+)^2$) which declares two pairs (a, b) and (a', b') to be related if the ratios of their coordinates are equal, that is, if $\frac{b}{a} = \frac{b'}{a'}$, which is equivalent to saying $a'b = ab'$. One easily verifies that this defines an equivalence relation on $(R^+)^2$. Note, for example, that $(2 : 3) = (4 : 6) = (\frac{1}{2} : \frac{3}{4})$. We denote the equvalence class of $(a, b) \in (R^+)^2$ by $(a : b)$, or sometimes just $a : b$, and we call it the *ratio* of a and b. Denoting the set of equivalence classes by $(\mathbb{R}^+ : \mathbb{R}^+)$, we see that the function

$$(3.1) \qquad \varphi : (\mathbb{R}^+ : \mathbb{R}^+) \to \mathbb{R}^+ \text{ defined by } \varphi((a : b)) = \frac{a}{b}$$

is well-defined, and that it is one-to-one and onto.

The Ratio Associated to an Interval. Since we have identified the set $\mathbb{R}^+$ with the set of pitches, or frequencies, the equivalence relation defined above applies to pairs of pitches (f_2, f_1), placing such a pair in an equivalence class $f_2 : f_1$, which is associated via φ to the number $r = \frac{f_2}{f_1}$ which we also call the ratio of f_2 to f_1. This number r is a measurement of the interval from the pitch f_1 to the pitch f_2. We will refer to both r and the corresponding class $f_2 : f_1$ as the *interval,* or *interval ratio,* determined by the frequencies f_1 and f_2. Thus each $r \in \mathbb{R}^+$ gives a unique interval. It is an enlightening exercise to listen to the intervals determined by various numbers, such as 3, 3/2, $\sqrt{2}$, 0.7, and even the transcendental numbers $\pi \approx 3.14159$ and $e \approx 2.71828$.

Orientation of Intervals. Intervals have an upward or downward orientation. We say that the interval given by pitches (f_2, f_1) (which we read as the interval from f_1 to f_2) is *upward* if $f_2 > f_1$ and *downward* if $f_2 < f_1$. In the former case we have $\frac{f_2}{f_1} > 1$; in the

latter case $\frac{f_2}{f_1} < 1$. Thus the upward intervals are given by the real numbers x which are greater than 1, and the downward intervals are given by the positive real numbers x which are less than 1.

$$\text{set of downward intervals} = \{x \in \mathbb{R} \mid 0 < x < 1\} = (0, 1)$$
$$\text{set of upward intervals} = \{x \in \mathbb{R} \mid 1 < x\} = (0, \infty)$$

The interval created when $f_1 = f_2$ will here be called the *unison interval.* It is given by the ratio $f : f$ (for any $f \in \mathbb{R}^+$), whcich corresponds via φ to the number 1.

Each interval $f_2 : f_1$ has a unique *opposite interval,* given by the ratio $f_1 : f_2$. It is the interval having the same "distance" in the opposite direction: if $f_2 : f_1$ is upward then $f_1 : f_2$ is downward, and vice-versa. If r is the real number ratio of an interval, then it's opposite interval has ratio r^{-1}.

If the orientation of an interval is not stated, it will be assumed that an upward interval is meant. For example if we say "the interval of a fourth" will be taken to mean "the upward interval of a fourth".

Multiplicativity. Observe that intervals have the following multiplicative property: If x_1 represents the interval $(f_2 : f_1)$ and x_2 the interval $(f_3 : f_2)$, then $x_1 x_2$ represents the interval $(f_3 : f_1)$. This is obvious, since $x_1 x_2 = \frac{f_2}{f_1}\frac{f_3}{f_2} = \frac{f_3}{f_1}$. Thus the result of juxtaposing two intervals, i.e., following one interval by another, is given by multiplying the two corresponding real numbers.

Multiplicative and Additive Measurements. The measurement of intervals by ratio is called *multiplicative,* because of the property stated above. The usual measurements of intervals, such semitones, steps, or octaves, are called *additive* because when we juxtapose two intervals we think of adding or subtracting. For example we say that 2 semitones plus 3 semitones equals 5 semitones; a fifth is a major third plus a minor third; a semitone is a major sixth minus a minor sixth. We will show later how this more conventional notion of interval relates to the multiplicative notion of an interval as a ratio.

Semitones. The principle of multiplicativity allows us to determine which real number which gives the interval of a semitone. Let us denote the number by s. Since twelve iterations of this interval gives the octave, which has ratio 2, we must have $s^{12} = 2$, which says (since s is positive)
$$s = \sqrt[12]{2} = 2^{1/12}\,.$$

If we iterate this interval n times to get n semitones, the ratio will be $\left(2^{1/12}\right)^n = 2^{n/12}$. It is natural to extend this conversion formula to an interval measured in semitones x, for any $x \in \mathbb{R}^+$:

(3.2) $\boxed{\text{The interval of } x \text{ semitones has ratio } 2^{x/12}.}$

This just follows from fact that $\left(2^{1/12}\right)^n = 2^{n/12}$.

Examples. The interval of a major third (4 semitones) has the ratio $2^{4/12} = 2^{1/3} = \sqrt[3]{2} \approx 1.25992$. The interval of downward a minor third (-3 semitones) has ratio $2^{-3/12} = 2^{-1/4} = 1/\sqrt[4]{2} \approx 0.840896$.

Frequencies of Chromatic Notes. If a note N has frequence f and an interval has ratio r, the note which lies the interval r from N has frequency rf. Given that A_4 is tunned to 440 Hz, we can now use a calculator to obtain the frequency of any other note on the keyboard.

For example, using the above calculation of the major third's ratio as $2^{1/3}$, we calculate in hertz the frequency f of $C_4^\sharp$, which lies a major third above A_3.

Since A_3 has frequency 220 Hz (being one octave below A_4) we have

$$f = 220 \cdot 2^{1/3} \approx 277.18\,.$$

Therefore $C_4^\sharp$ should be tuned to 277.18 Hz.

Microtuning and Cents. We will see later that mathematical tuning involves intervals which cannot be as an integer multiple of semitones. The term *microtuning* refers to systems of tuning which alters the frequencies of notes in the equally tempered chromatic scale, or adds new notes to that scale.

For this the semitone is divided into 100 equal intervals, the subdivision being called a *cent*. Thus 1200 iterations of this interval gives an octave. The interval of one cent is so small as to be imperceptible to most of us. Even the interval of 10 cents is difficult to perceive. Therefore the measurement of intervals in cents is fine enough to be quite satisfactory for microtuning.

Cents, like semitones and octaves, is an additive measurement of intervals.

Conversion of Cents to a Ratio. Letting c denote the ratio corresponding to one cent, then by reasoning as we did with semitones, we have $c^{1200} = 2$, i.e.,

$$c = 2^{1/1200} \approx 1.0005778$$

For any number x (not necessarily an integer), the interval of x cents has the ratio r given by

$$r = c^x = \left(2^{1/1200}\right)^x = 2^{x/1200}\,.$$

Thus $r = 2^{x/1200}$ gives the conversion of x cents to a ratio r.

(3.3) $\boxed{\text{The interval of } x \text{ cents has ratio } 2^{x/1200}.}$

This relationship allows us to convert cents to a ratio using a scientific calculator. For example, the interval of 17 cents corresponds to the number $2^{17/1200} \approx 1.009868$.

Arbitrary Chromatic Units. Suppose n is a positive integer and we wish to divide the octave into n equal subintervals, which we will call n-*chromatic units.* The same reasoning that led to formulas (3.2) and (3.3) tells us that:

(3.3) $\boxed{\text{The interval of } x \text{ } n\text{-chromatic units has ratio } 2^{x/n}.}$

Octave Equivalence of Interval Ratios. By definition, two intervals are equivalent modulo octave if they differ by an interval of n octaves, for some $n \in \mathbb{Z}$. The difference of two intervals is the result of juxtaposing the first with the opposite of the second. If the intervals are given by ratios r_1 and r_2, this difference is given by the interval ratio $r_1 r_2^{-1}$. The interval of n octaves has ratio 2^n. Thus we have:

PROPOSITION. *Two interval ratios r_1 and r_2 are equivalent modulo octave if and only if there exists $n \in \mathbb{Z}$ such that $r_1 r_2^{-1} = 2^n$.*

For example, the interval ratios 41 and 328 are equivalent modulo octave, since $\frac{41}{328} = \frac{1}{8} = 2^{-3}$.

Conversion to Additive Measurements. We eventually will need to be able to convert the ratio measurement of a musical interval to an additive measurement such as cents or semitones. Suppose we are given a ratio r to convert to cents. In this situation we must solve for x in the equation $r = 2^{x/1200}$. This requires taking a *logarithm*, a topic which will be reviewed and developed in the next section. The following observation provides additional motivation for evoking logarithms. If we plot pitches on an axis according to their frequency, we see that musical intervals are not represented as distance along the axis. For example, the pitches A_2, A_3, A_4, and A_5 appear as:

The distances on the frequency axis between A_n, A_{n+1} for various integers n are different though the musical intervals are all the same – one octave. This is somewhat unsatisfying since we are used to thinking of two pairs of pitches representing the same interval as being the same "distance" apart, which is roughly the situation on a musical staff, where the vertical "distance" between each successive A, notated on the same clef, appears to be the same:
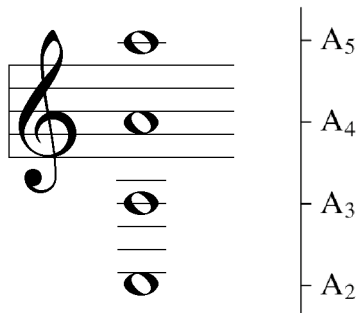
However if we plot the pitches according to the logarithms of their frequencies, we get a more satisfying result. The logarithm will also enable us to measure in semitones an interval expressed as a ratio r or given by two frequencies (f_1, f_2).

Vibration of Strings. It was known by the ancient Greeks that the vibrating frequency of a string is inversely proportional to the length of the string, provided the tension on the string and the weight of the string per unit of length remain the same. This says the relationship between length L and frequency F can be expressed as

$$(3.5) \qquad\qquad F = \frac{k}{L}$$

for some $k \in \mathbb{R}^+$.

Some stringed intruments have frets so that the length of the string, and hence the pitch it sounds when strummed, can be altered in performance. Let us consider how the fret can be positioned to effect a given change of frequency. Suppose the string has length L and its frequency is f. Visualize the string stretching horizontally, and suppose a fret is positioned at distance d from, say, the right end of the string.

We want to calculate the interval ratio $f' : f$, where f' is the frequency of the segment of string to the right of the fret. This segment has length $L' = L - d$, so by (3.5) we have

$$(f' : f) = (\frac{k}{L'} : \frac{k}{L}) = (\frac{1}{L'} : \frac{1}{L}) = (\frac{1}{L-d} : \frac{1}{L}),$$

which corresponds, via the function φ defined in (3.1) to the number

$$\varphi(f' : f) = \frac{\frac{1}{L-d}}{\frac{1}{L}} = \frac{L}{L-d}.$$

If d is expresses in terms of its proportion to L, i.e., $d = qL$, the fraction becomes

$$\frac{L}{L-qL} = \frac{L}{L(1-q)} = \frac{1}{1-q}.$$

Putting this together, we have

$$(3.6) \qquad\qquad \frac{f'}{f} = \frac{1}{1-q}$$

Suppose we want to place a fret so as to move the pitch upward by a specified ratio $r \geq 1$. This means $r = \frac{f'}{f}$, so by (3.6) we solve for q in the equation $\frac{1}{1-q} = r$ to get $1 - q = r^{-1}$, or

$$(3.7) \qquad\qquad\qquad \boxed{q = 1 - r^{-1}}$$

Example. Suppose we want to place a fret so as to move the pitch upward a major third above f. Since the major third has ratio $r = 2^{1/3}$, we have $q = 1 - \left(2^{1/3}\right)^{-1} = 1 - 2^{-1/3} \approx 0.206299$. The position of the fret is $(0.206299)L$, which is close to $\frac{1}{5}L$.

Exercises

(1) Express each of these intervals as a radical or the reciprical of a radical, as a power of 2, and by a decimal approximation with 2 decimal digits:

      (a) up 32 cents
      (b) down 750 cents
      (c) up a minor third
      (d) the interval from C_3 to $F_1^\sharp$

(2) Assuming A_4 is tuned to 440 Hz, find the frequencies for all the notes of the chromatic scale from C_4 to C_5.

(3) Suppose middle C is tuned as 256 Hz. Find the frequencies for:

      (a) A_4 (b) $D_2^\sharp$ (c) C_3 (d) $F_1^\sharp$

(4) For each of these chords, voiced within an octave with the root on the bottom, give the pitch of each note in the chord:

      (a) major triad with root C_3
      (b) minor triad with root $G_4^\sharp$
      (c) minor seventh chord with root A_5
      (d) diminished triad with root $B_4^\flat$

(5) Suppose a string on a banjo has length 50 cm. Indicate positions of the 12 frets which will allow the string to play one octave of the ascending chromatic scale.

(6) Determine whether each pair of interval ratios are are equivalent modulo octave.

      (a) 5, 20   (b) 14, $\frac{7}{2}$ (c) 2.3, 9.2   (d) 1.04, 0.13   (e) $\pi, \frac{3\pi}{2}$