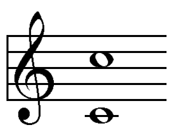
CHAPTER VIII

ALGEBRAIC PROPERTIES OF THE INTEGERS

We have identified a musical interval I with a positive real number $x \in \mathbb{R}^+$. Since $\mathbb{Z}^+ \subset \mathbb{R}^+$, each positive integer gives an interval. For example, we have seen that the integer 2 represents the octave, and that the integer 3 is an interval about 2 cents greater than the keyboard's octave-and-a-fifth (1900 cents), as shown by the calculation $1200 \log_2 3 \approx 1901.96$.



$2 =$ octave interval

$3 \approx$ octave-and-a-fifth interval

$4 =$ two octave interval

We will now investigate some properties of the integers $\mathbb{Z}$ which relate to musical phenomena.

Ring. A non-empty set R endowed with two associative laws of composition $+$ and $\cdot$ is called a *ring* if $(R, +)$ is a commutative group, $(R, \cdot)$ is a monoid, and for any $a, b, c \in R$ we have $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ (The latter property is called *distributivity.*). We call the $+$ operation *addition* and the $\cdot$ operation *multiplication*, and we often denote the latter by dropping the $\cdot$ and simply writing ab for $a \cdot b$. We write 0 and 1 for the additive and multiplicative identity elements, respectively. We say the ring R is *commutative* if the monoid $(R, \cdot)$ is commutative. (We have already insisted that $(R, +)$ is commutative.) We will be dealing only with commutative rings here, so henceforth when we say "ring" we will mean "commutative ring".

Two properties that we would expect to hold for any x in a ring R are these: $(-1) \cdot x = -x$ and $0 \cdot x = 0$. We leave it as an exercise that these properties can indeed be deduced from our assumptions.

Units. We have assumed that $(R, \cdot)$ is a monoid; it will not be a group in general[1] since 0 has no multiplicative inverse. However, some elements of R (1, for example) will have multiplicative inverses. If $x \in R$ is such an element, we call x a *unit,* and we denote its multiplicative inverse[2] is by x^{-1}. The set of units in R, sometimes denoted R^*, form a

[1] The only situation when $(R, \cdot)$ is a group is when $R = \{0\}$, which coincides with the case $0 = 1$. In this case R is called the *trivial ring.*

[2] The multiplicative inverse x^{-1} is unique to x. The proof of this mimics the proof that inverses in a group are unique.

group with respect to multiplication.

Cancellation. A ring R is called an *integral domain* if whenever $a, b \in R$ with $ab = 0$, then $a = 0$ or $b = 0$.

PROPOSITION (CANCELLATION). *If R is an integral domain, and $a, b, c \in R$ with $a \neq 0$ and $ab = ac$, then $b = c$.*

PROOF. We have $0 = ab - ac = a(b - c)$. Since $a \neq 0$ and R is an integral domain, we must have $b - c = 0$, i.e., $b = c$.

Examples. The reader should verify the details in the following four examples.

(1) **Integers.** The set of integers $\mathbb{Z}$, taking $+$ and $\cdot$ to be the usual addition and multiplication, is the most basic example of a ring. It is commutative, and it is an integral domain. The group of units is $\mathbb{Z}^* = \{1, -1\}$.

(2) **Real Numbers.** The set $\mathbb{R}$ also becomes a ring under the usual $+$ and $\cdot$. It is also an integral domain. Here we have $\mathbb{R}^* = \mathbb{R} - \{0\}$.

(3) **Rational Numbers.** $\mathbb{Q}$ is an integral domain, sharing with $\mathbb{R}$ the property that all non-zero elements are units.

(4) **Modular Integers.** For $m \in \mathbb{Z}^+$, we give $\mathbb{Z}_m$ a ring structure as follows: The additive group $(\mathbb{Z}_m, +)$ is as before. For $[k], [\ell] \in \mathbb{Z}_m$, define $[k] \cdot [\ell] = [k\ell]$. The proofs that this is well defined and that the axioms for a ring are satisfied by $+$ and $\cdot$ are left as an exercise. Note that $[0]$ and $[1]$ are the additive and multiplicative identity elements, respectively, of $\mathbb{Z}_m$.

Ideals. A subset $J \subseteq R$ ic called an *ideal* if it is a subroup of the additive group $(R, +)$ and if whenver $a \in R$ and $d \in J$, then $ad \in J$.

One example of an ideal in R is the *zero ideal* $\{0\}$. Any other ideal will be called a *non-zero ideal.* The ring R itself is an ideal.

Given $a \in R$ we can form the set of all multiples of a in R, namely the set

$$aR = \{x \in R \,|\, x = ab \text{ for some } b \in R\}.$$

Such an ideal is called a *principal ideal*, and the element a is called a generator for the ideal. Note that $\{0\}$ and R are principle ideals by virtue of $\{0\} = 0R$ and $R = 1R$.

If R is an integral domain in which every ideal is principal, we call R a *principal ideal domain,* abbreviated PID.

For example, the set of even integers forms an ideal in $\mathbb{Z}$. This ideal is a principal ideal, since it is equal to $2\mathbb{Z}$. We will now show that:

THEOREM. *$\mathbb{Z}$ is a principal ideal domain.*

PROOF. This is based on the Euclidean algorithm. Let J be an ideal in $\mathbb{Z}$. If $J = \{0\}$, then $J = 0\mathbb{Z}$ and we are done. Otherwise J contains non-zero integers, and since $n \in J$

implies $(-1)n = -n$ is in J, then J must contain some positive integers. Let n be the smallest positive integer in J (such an n exists by the well ordering principle). We claim that $J = n\mathbb{Z}$. Clearly $n\mathbb{Z} \subseteq J$. To see the other containment, let $m \in J$, and use the Euclidean algorithm to write $m = qn + r$ with $0 \leq r < n$. Then r is in J since $r = m - qn$. By the minimality of n, we conclude $r = 0$, hence $n = qn \in b\mathbb{Z}$ as desired.

If $J \subseteq \mathbb{Z}$ is an ideal with $J \neq 0$, and if n is a generator for J, then the only other generator for J is $-n$. This follows easily from the fact that any two generators are multiples of each other, and will be left as an exercise. Thus any non-zero ideal has a unique positive generator.

Greatest Common Divisor. Given $m, n \in \mathbb{Z}$, We note that the subset $m\mathbb{Z} + n\mathbb{Z}$, by which we mean the set of all integers a which can be written $a = hm + kn$ for some $h, k \in \mathbb{Z}$, is an ideal in $\mathbb{Z}$. Therefore it has a unique positive generator d, which divides both m and n. If e is any other positive integer which divided both m and n then $m, n \in e\mathbb{Z}$ so $m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z} \subseteq e\mathbb{Z}$, and hence e divides d. Therefore $d \geq e$ and we (appropriately) call d the *greatest common divisor* of m and n. The greatest common divisor is denoted $\gcd(m, n)$. Since $d\mathbb{Z} = m\mathbb{Z} + n\mathbb{Z}$, there exist integers h, k such that $d = hm + kn$.

To say that $\gcd(m, n) = 1$ is to say that the only common divisors of m and n in $\mathbb{Z}$ are ± 1. In this case we say that m and n are *relatively prime.*

Prime Numbers. A positive integer p is called *prime* if it is divisible in $\mathbb{Z}$ by precisely two positive integers, namely 1 and p. (Note that 1 is not prime by virtue of the word "precisely".) The first ten prime numbers are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29 \tag{1}$$

It will be left as an exercise to show that if p is prime and $n \in \mathbb{Z}$, then either p divides n or $\gcd(p, n) = 1$.

Sieve of Eratosthenes. A systematic procedure for finding the prime numbers was given by the Greek astronomer/mathematician Eratosthenes of Cyrene (3rd century BC). We conceive of the positive integers as an infinite list $1, 2, 3, 4, 5, 6, \ldots$, then proceed to cross out certain numbers on the list, as follows. After crossing out 1, we cross out all numbers following 2 which are divisible by 2.

$$\cancel{1}, 2, 3, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, 9, \cancel{10}, 11, \cancel{12}, 13, \cancel{14}, 15,$$
$$\cancel{16}, 17, \cancel{18}, 19, \cancel{20}, 21, \cancel{22}, 23, \cancel{24}, 25, \cancel{26}, 27, \cancel{28}, 29, \cancel{30}, \ldots$$

Then we find the next number after 2 which is still on the list, which is 3. We then cross out all numbers following 3 which are not divisible by 3.

$$\cancel{1}, 2, 3, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, \cancel{9}, \cancel{10}, 11, \cancel{12}, 13, \cancel{14}, \cancel{15},$$
$$\cancel{16}, 17, \cancel{18}, 19, \cancel{20}, \cancel{21}, \cancel{22}, 23, \cancel{24}, 25, \cancel{26}, \cancel{27}, \cancel{28}, 29, \cancel{30}, \ldots$$

When this process can be continued up to an integer n, the the numbers below n which remain on the list are precisely the primes which are $\leq n$.

$$\cancel{1}, 2, 3, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, \cancel{9}, \cancel{10}, 11, \cancel{12}, 13, \cancel{14}, \cancel{15},$$
$$\cancel{16}, 17, \cancel{18}, 19, \cancel{20}, \cancel{21}, \cancel{22}, 23, \cancel{24}, \cancel{25}, \cancel{26}, \cancel{27}, \cancel{28}, 29, \cancel{30}, \ldots$$

We have shown that the primes ≤ 30 are the ten integers in the list (1) above.

If the procedure were continued infinitely to completion, the complete list of primes would remain.

THEOREM. *If p is a prime number and if p divides mn, where $m, n \in \mathbb{Z}$, then p divides m or p divides n.*

PROOF. Suppose p does not divide m. Then $\gcd(m, p) = 1$ and we can write $1 = hm + kp$ for some integers h and k. Multiplying this equation by n gives $n = hmn + kpn$. Note that p divides both summands on the right, since p divides nm. therefore p divides n. This concludes the proof.

One can easlily conclude that if a prime number p divides a product $m_1 m_2 \cdots m_s$, then p divides at least one of $m_1, m_2, \ldots, m_s$.

Unique Factorization. We now establish the fact that every positive integer can be factored uniquely as the product of primes.

THEOREM. *Let $n \geq 1$ be an integer. Then n can be factored as*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

where $r \geq 0$, $p_1, p_2 \ldots, p_r$ are distinct primes, and $\alpha_1, \alpha_2, \ldots, \alpha_r \geq 1$. Moreover, this factorization is unique, meaning that if $n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_t^{\beta_t}$ is another such factorization, then $t = r$ and after rearranging we have $p_1 = q_1$, $p_2 = q_2$, $\ldots$, $p_r = q_r$.

PROOF. We first establish the existence of a prime factorization for all integers ≥ 1. If not all positive integers admit a prime factorization, then by the Well-Ordering Principle we can choose a smallest integer n which fails to admit a factorization. We note that n itself could not be prime, otherwise it admits the factorization in the theorem with $r = 1$ and $p_1 = n$. Since n is not prime, it has a positive divisor m which is neither n nor 1. We have $n = m\ell$ and clearly ℓ is neither n nor 1. We must have $1 < m, \ell < n$, so by the minimality of n, both m and ℓ have prime factorizations. But if m and ℓ have prime factorizations, then so does n since $n = m\ell$. This is a contradiction. Hence all integers ≥ 1 have a prime factorization.

It remains to show the uniqueness. If $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} = q_1^{\beta_1} q_2^{\beta_2} \cdots q_t^{\beta_t}$, then p_1 divides $q_1^{\beta_1} q_2^{\beta_2} \cdots q_t^{\beta_t}$. Since p_1 is prime it must divide one of $q_1, q_2, \ldots, q_t$. Say p_1 divides q_1. Since q_1 is also prime we must have $p_1 = q_1$, so we can cancel to get $p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} = p_1^{\beta_1 - 1} q_2^{\beta_2} \cdots q_t^{\beta_t}$. We continue cancelling p_1 to deduce that $\alpha_1 = \beta_1$. The remaining equation is $p_2^{\alpha_2} \cdots p_r^{\alpha_r} = q_2^{\beta_2} \cdots q_t^{\beta_t}$. As above we can argue that $p_2 = q_2$ (after rearranging) and that $\alpha_2 = \beta_2$. We continue to get the desired result.

Modular Integers. The algebraic properties we have established for $\mathbb{Z}$ tell us many things about the rings of modular integers $\mathbb{Z}_m$, for $m \in \mathbb{Z}^+$. One such fact concerns the matter of when an element $[n] \in \mathbb{Z}_m$ is a generator of the additive group $(\mathbb{Z}_m, +)$.

THEOREM. *Given $[n] \in \mathbb{Z}_m$, the following three conditions are equivalent.*

(1) *$gcd(m, n) = 1$.*
(2) *$[n]$ is a generator of the additive group $(\mathbb{Z}_m, +)$.*
(3) *$[n]$ is a unit in the ring $\mathbb{Z}_m$ (i.e., $[n] \in \mathbb{Z}_m^*$).*

PROOF. We first consider conditions (2) and (3). If $[n]$ is a generator of $(\mathbb{Z}_m, +)$, then all elements of $\mathbb{Z}_m$ can be written as $k \cdot [n]$, for some $k \in \mathbb{Z}$. (This is the way we write exponentiation in an additive group.) In particular, we have $[1] = k \cdot [n]$. But, by the definition of multiplication in $\mathbb{Z}_m$, $k \cdot [n] = [k] \cdot [n]$. Therefore $[k] \cdot [n] = [1]$, which shows $[n]$ is a unit. Conversely, if $[n] \in \mathbb{Z}_m^*$, with inverse $[k] = [n]^{-1}$, then for any $[\ell] \in \mathbb{Z}_m$ we have $[\ell] = [\ell] \cdot [1] = [\ell] \cdot [k] \cdot [n] = [\ell k] \cdot [n] = \ell k \cdot [n]$, which shows that $[\ell]$ is a multiple ("power") of $[n]$. Hence $[n]$ is a group generator for $(\mathbb{Z}_m, +)$.

The equivalence of (1) with these conditions, the proof of which uses greatest common divisors, is left as an exercise.

Euler Phi Function. For any $m \in \mathbb{Z}^+$, we have defined the *Euler phi function* $\phi(m)$ to be the number of positive integers n with $1 \leq n < m$ which are relatively prime to m. According to the above theorem, $\phi(m)$ also counts the number of elements in $\mathbb{Z}_m^*$, and the number of group generators for $(\mathbb{Z}_m, +)$. By virtue of the latter, $\phi(m)$ counts the number of generating intervals in the m-chromatic scale.

For example $\phi(12) = 4$, since the numbers $1, 5, 7, 11$ are precisely the positive integers ≤ 12 which are relatively prime to 12. This reflects the fact that the generating intervals in the 12-chromatic scale are the semitone, the fourth, the fifth, and the major seventh.

Exercises

(1) Prove that in any (commutative) ring R we have $(-1) \cdot x = -x$ and $0 \cdot x = 0$, for any $x \in R$.

(2) Give the prime factorizations of these integers, writing the primes in ascending order, as in $2^3 \cdot 3 \cdot 7^2$.

(a) 110 (b) 792 (c) 343 (d) 3422 (e) 15×10^{23}

(3) Call a musical interval a *prime interval* if its interval ratio is a prime integer; call it a *rational interval* if its interval ratio is a rational number. Show that all rational intervals can be written as compositions of prime intervals and their opposites.

(4) Express each of these ideals in $\mathbb{Z}$ in the form $n\mathbb{Z}$, where n is a positive integer:

(a) $12\mathbb{Z} + 15\mathbb{Z}$ (b) $5\mathbb{Z} + (-20)\mathbb{Z}$

(c) $10\mathbb{Z} + 44\mathbb{Z}$ (d) $13\mathbb{Z} + 35\mathbb{Z}$

(5) Verify that $\mathbb{Q}$ (the rational numbers) is a ring, and, in fact, an integral domain. Show that the only ideals in $\mathbb{Q}$ are $\{0\}$ and $\mathbb{Q}$.

(6) Prove that there are infinitely many prime numbers. (Hint: If $p_1, \ldots, p_n$ were a complete list of primes, consider a prime factor of $p_1 \cdots p_n + 1$.)

(7) Prove that if p is prime and $n \in \mathbb{Z}$, then either $p \mid n$ or $\gcd(p, n) = 1$.

(8) Given $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, prove that $[n]$ is a generator for $\mathbb{Z}_m$ if and only if $\gcd(m, n) = 1$. Interpret this as a statement about generating intervals in the modular m-chromatic scale.

(9) Prove that m iterations of any m-chromatic interval is a multioctave. Interpret this as a statement about an element $[k]$ of $\mathbb{Z}_m$, and use this statement to prove that the order r of $[k]$ divides m.