

## CHAPTER IX

### THE INTEGERS AS INTERVALS

We will now determine, for each of the first several positive integers  $n = 1, 2, 3, \dots$ , which tempered scale interval best approximates the interval given by the ratio  $n$  and we will calculate the closeness of the approximation. This will tell us how to detune keyboard intervals so that the integer ratios can be heard. Once this is done, it is enlightening to “listen to the integers”, noting that each possesses a unique “personality” which seems determined by the integer’s prime factorization.

We will occasionally employ the slightly awkward term *integral interval* to refer to a musical interval whose ratio is an integer. We call such an interval a *prime interval* if its ratio is a prime.

The set of integral intervals forms a monoid under composition of intervals; this monoid can be identified with  $(\mathbb{Z}, \cdot)$ .

**One.** The ratio 1, representing *unison*, is the identity element of the monoid  $(\mathbb{Z}, \cdot)$  of integral intervals, and the identity element of the group  $(\mathbb{R}, \cdot)$  of all interval ratios. It is not terribly interesting, since it is the ratio of two frequencies giving the same pitch.

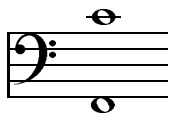
**Two.** We have noted the fact that first prime, 2, gives the *octave*, which might be called music’s most consonant interval. When two notes an octave apart are sounded they blend together almost as one. Octave equivalence is ingrained in musical notation by virtue of the fact that notes which form the interval of one or more octaves are assigned the same letter of the alphabet. Only by using subscripts such as  $C_2$  or  $A_5^b$  (or by using a musical staff) can we distinguish them notationally.

Moreover, the keyboard’s equally tempered chromatic scale is tuned to give a perfect octave (since equal temperament is obtained by dividing the interval given by 2 into 12 equal intervals). Hence the ratio 2 is rendered precisely by equal temperament. The interval from  $F_2$  to  $F_3$ , shown below, has frequency ratio exactly 2.



keyboard’s exact representation of 2

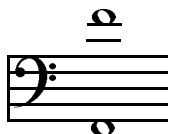
**Three.** We have noted that the prime interval 3 is best approximated on the keyboard by 19 semitones, or one octave plus a fifth, shown below as the interval from  $F_2$  to  $C_4$ .



keyboard's approximation of 3,  $\approx 2$  cents flat

This approximation is about 2 cents under, since 3 is measured in cents by  $1200 \log_2 3 \approx 1901.96$ , and 1900 cents is 19 semitones, which is an octave plus a fifth. This is a very good approximation; it is very difficult for most of us to perceive the difference between the octave plus a fifth and the interval given by 3.

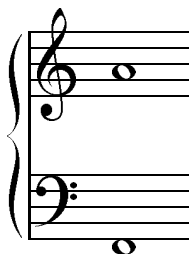
**Four.** The ratio 4 is two octaves by virtue of  $4 = 2^2$ . It can be played precisely on the keyboard, as can any integral ratio which is a power of two.



keyboard's exact representation of 4

We will see that the powers of 2 are the only positive integers which is can be played perfectly on a keyboard tuned to the the 12-note equally tempered scale, or in fact on any chromatic scale which equally divides the octave. Yet we will see that harmony derives from the integers.

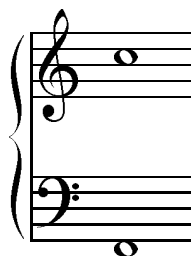
**Five.** The next interesting integer ratio is the prime number 5, which is given in cents by  $1200 \log_2 5 \approx 2786.31$ . The closest interval to this on the keyboard is 2800 cents, which is two octaves plus a major third.



keyboard's approximation of 5,  $\approx 14$  cents sharp

This is sharp by about 14 cents. Unlike the fifth's approximation of 2, this difference is perceptible, upon a careful listening, by most people with reasonably good pitch discrimination. The tempered scale was shunned many years primarily because of this particular discrepancy.

**Six.** The integer  $6 = 3 \cdot 2$  is the smallest integer whose prime factorization involves more than one prime. By virtue of the factorization  $6 = 2 \cdot 3$ , multiplicativity tells us that this interval is obtained by iterating the intervals corresponding to 2 and 3. Thus we get an interval which is approximated on the keyboard by an octave plus an octave plus a fifth, or two octaves and a fifth.



keyboard's approximation of 6,  $\approx 2$  cents flat

Since the keyboard renders the octave precisely, its rendition of six should have the same error as its approximation of 3, which is about 2 cents. This is verified by the calculation

$$\begin{aligned} 1200 \log_2 6 &= 1200(\log_2 2 + \log_2 3) \\ &= 1200 \log_2 2 + 1200 \log_2 3 \\ &\approx 1200 + 1901.96 = 3101.96 \end{aligned}$$

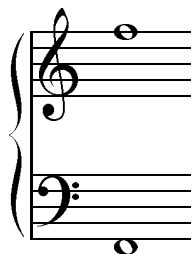
which shows the ratio 6 to be about 2 cents greater than 3100 cents (= 31 semitones), which is the keyboard's two octaves plus a fifth.

**Seven.** The prime 7 is the lowest integer which is poorly approximated by the tempered chromatic scale. In cents it is given by  $1200 \log_2 7 \approx 3368.83$ . The closest interval on the keyboard is 3400 cents, which over-estimates 7's interval by about 31 cents. This approximation is 34 semitones, which equals two octaves plus a minor seventh.



keyboard's approximation of 7,  $\approx 31$  cents sharp

**Eight.** Continuing, we note that 8, being  $2^3$ , is exactly three octaves, and is rendered precisely on the keyboard.



keyboard's exact representation of 8

**Nine.** Since  $9 = 3^2$ , it is approximated by iterating the octave-plus-a-fifth interval with

itself, which yields two octaves plus a ninth, or three octaves plus a step. This has double the error of the approximation of 3, so the approximation of 9 is about 4 cents flat.



keyboard's approximation of 9,  $\approx$  4 cents flat

**Ten.** We have  $10 = 2 \cdot 5$ , hence 10 is approximated by the composition of the octave with the two-octaves-plus-a-third interval, yielding three octaves and a third, and having the same error as the approximation of 5 (since 2 is rendered exactly), which is about 14 cents sharp.



keyboard's approximation of 10,  $\approx$  14 cents sharp

**Eleven.** The next integer, the prime 11, has the worst tempered scale approximation encountered so far:  $1200 \log_2 11 \approx 4151.32$ . Notice that it lies very close to halfway between 41 semitones (three octaves plus a fourth) and 42 semitones (three octaves plus a tritone), slightly closer to the latter.



keyboard's approximation of 11,  $\approx$  49 cents sharp

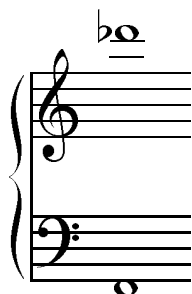
This interval is truly “in the cracks”, lying about a quarter step from the closest tempered scale intervals.

**Twelve.** We note that 12, being  $2^2 \cdot 3$ , is approximated 14 cents sharp by three octaves plus a fifth.



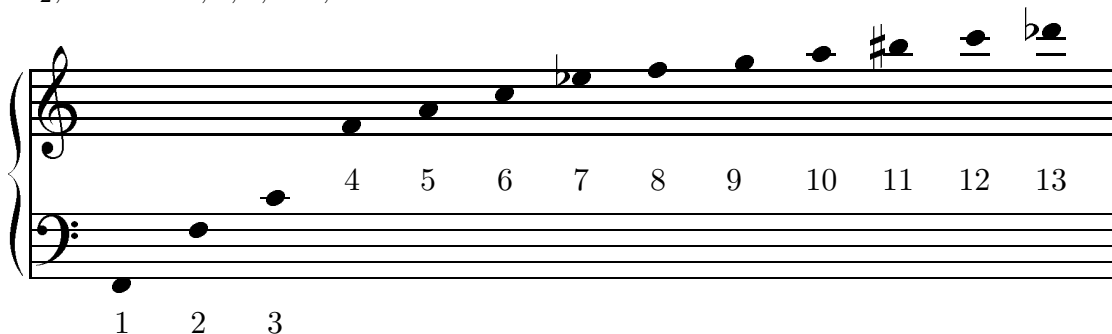
keyboard's approximation of 12,  $\approx 2$  cents flat

**Thirteen.** The last integer we will consider here is the prime 13. Since  $1200 \log_2 13 \approx 4440.53$ . Therefore 13 is best approximated on the keyboard by 44 semitones, or three octaves plus a minor sixth, and the approximation is about 41 cents flat.



keyboard's approximation of 13,  $\approx 41$  cents flat

**Summary.** The sequence of chromatic notes best approximating the pitch having ratio  $n$  with  $F_2$ , for  $n = 1, 2, 3, \dots, 13$  is:



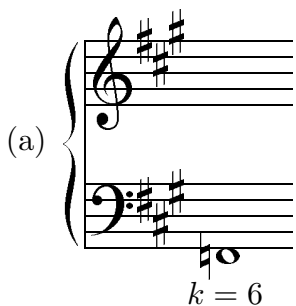
**Non-Chromatic Nature of Intervals Other Than Multiple Octaves.** Note that the only integral intervals on the keyboard so far are the powers of 2 (multiple octave intervals). The following theorem shows that no other integer ratios  $n$  occur on the keyboard.

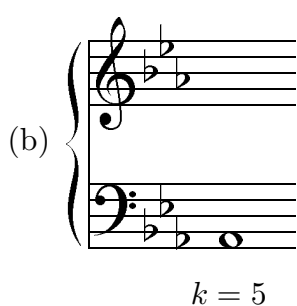
**THEOREM.** *The only keyboard intervals which have integer ratios are the powers of 2.*

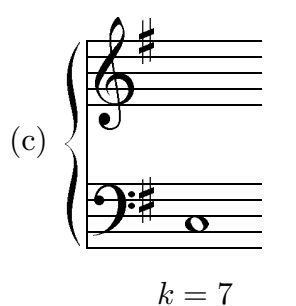
**PROOF.** Suppose  $n \in \mathbb{Z}^+$  is a keyboard interval. This means it is obtained by composing  $k$  semitones, for some integer  $k \geq 0$ . Since the semitone has interval ratio  $2^{1/12}$ , we have  $n = (2^{1/12})^k = 2^{k/12}$ . Raising this to the power 12, we get  $n^{12} = 2^k$ . By the unique factorization theorem,  $n$  can have only 2 in its prime factorization.

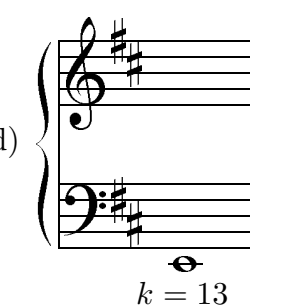
### Exercises

- (1) For each given note  $N$  and integer  $k$ : label  $N$  by letter and subscript (e.g.,  $A_4^b$ ); write on the staff the (12-chromatic) note  $M$  which best approximates the pitch having interval ratio  $k$  with  $N$ ; and label  $M$  by letter and subscript.

(a)   $k = 6$

(b)   $k = 5$

(c)   $k = 7$

(d)   $k = 13$

- (2) For each given pair of positive integers  $m$  and  $n$ , express in  $m$ -chromatic units the  $m$ -chromatic scale's best approximation of the integer ratio  $n$ , and indicate how many cents sharp or flat the approximation is.

- (a)  $m = 7, \quad n = 5$   
 (b)  $m = 19, \quad n = 3$   
 (c)  $m = 13, \quad n = 7$   
 (d)  $m = 4, \quad n = 8$   
 (e)  $m = 23, \quad n = 11$

- (3) List the primes larger than 13 but less than 50, and for each, determine how closely its musical interval is approximated by the keyboard, calculating the error in cents.