CHAPTER X

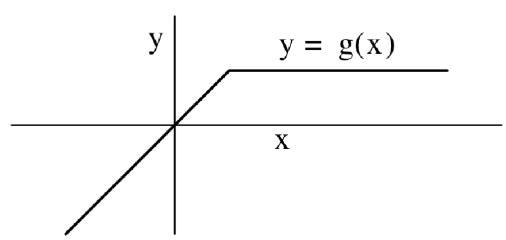
TIMBRE AND PERIODIC FUNCTIONS

Timbre. The term *timbre* refers to the quality or distinguishing properties of a musical tone other than its pitch, i.e., that which enables one to distinguish between a violin, a trombone, a flute, the vowel ō, or the vowel ē, even though the tones have the same pitch. In order to address this phenomenon we need to discuss a few more concepts relating to functions and graphs.

Piecewise Definitions and Continuity. A function can be defined in piecewise fashion, for example,

$$g(x) = \begin{cases} x, & \text{for } x \le 1 \\ 1, & \text{for } x > 1, \end{cases}$$

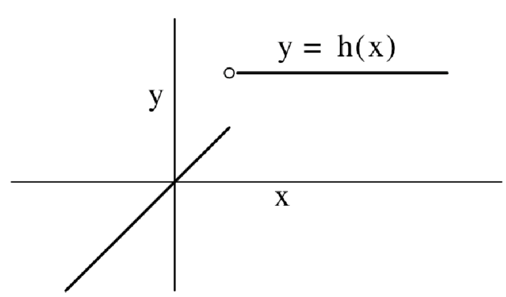
whose graph is:



or

$$h(x) = \begin{cases} x, & \text{for } x \le 1 \\ 2, & \text{for } x > 1, \end{cases}$$

whose graph is:



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Note the "jump" that appears in the graph of $y = h(x)$ at $x = 1$. This is an example of a *discontinuity,* i.e., the situation at a point $x = a$ at which the function fails to be continuous, as per the following definition.

DEFINITION. A function $y = f(x)$ is defined to be *continuous* at $x = a$ if given any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

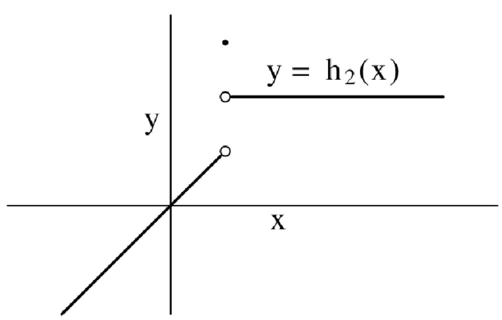
The function

$$h_1(x) = \begin{cases} x, & \text{for } x < 1 \\ 2, & \text{for } x \geq 1 \end{cases}$$

has the same graph as $h(x)$ except at $x = 1$. We could assign $f(1)$ to be some other number, as in

$$h_2(x) = \begin{cases} x, & \text{for } x < 1 \\ 3, & \text{for } x = 1 \\ 2, & \text{for } x > 1, \end{cases}$$

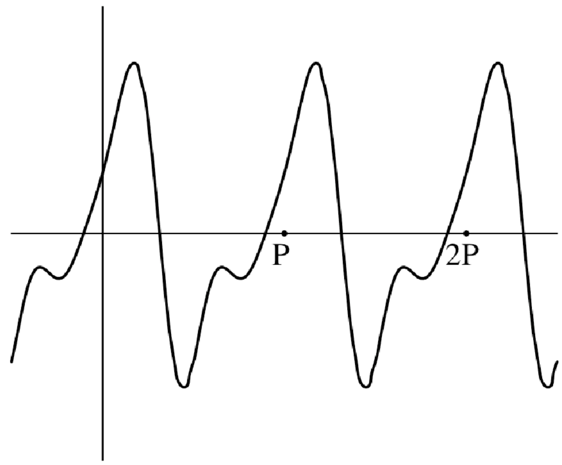
which has the graph



which again has a discontinuity at $x = 1$. It is not hard to prove that there is, in fact, no way to reasign $h(1)$, leaving all other values of h unchanged, in such a way that h is continuous at $x = 1$.

A rough interpretation of a discontinuity is a "jump" in the graph. (This is not precise mathematical terminology, but it serves us pretty well intuitively.) A function which is continuous on an interval I is one whose graph has no "jumps" for any $x \in I$.

Periodic Functions. A function $f(x)$ whose domain is all of $\mathbb{R}$ is called *periodic* if there is a positive number P such that for all $x \in \mathbb{R}$, $f(x + P) = f(x)$. This means that the behavior of the function is completely determined by its behavior on the half-open interval $[0, P)$ (or on <u>any</u> half-open interval of width P).



The number P is called the *period* of the function.

Example. The functions $y = \sin x$ and $y = \cos x$ are periodic of period 2π.

Any function $f(x)$ defined on the interval $[0, P)$ can be extended (uniquely) to a periodic function $g(x)$ of period P whose domain is all of $\mathbb{R}$. This is done by setting $g(x) = f(x - nP)$ for $x \in [nP, (n+1)P)$ for all integers n.

Effect of Shifting and Stretching on Periodicity. If $y = f(t)$ is a periodic function with period P, then the vertical and horizontal shifts $y = f(t) + c$ and $y = f(t - c)$, for $c \in \mathbb{R}$ are also periodic of period P, as is the vertical stretch $y = cf(t)$. However the horizontal stretch $y = f(t/c)$ will have period cP. So the effect of stretching horizontally by a factor of c is to divide the the frequency of $f(t)$ by c. The proofs of these assertions will be left as an exercise.

Shifting and Stretching Sine and Cosine. The graph of $y = \cos x$ is obtained by shifting the graph of $y = \sin x$ to the left by $c = \frac{\pi}{2}$. This is because the sine and cosine funtions have the relationship

$$\cos x = \sin\left(x + \frac{\pi}{2}\right),$$

which is a special case of the "summation" formula

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,. \tag{1}$$

Note that the former equation is obtained from the latter by setting $\alpha = x$ and $\beta = \frac{\pi}{2}$, since $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

More generally, if we treat (1) as a functional equation by replacing α by the independent variable x and letting β be some fixed number (we might wish to think of β as being an angle measured in radians), we have

$$\sin(x + \beta) = \cos\beta\sin x + \sin\beta\cos x\,. \tag{2}$$

The numbers $\cos\beta$ and $\sin\beta$, are the coordinates of the point Q on the unit circle (i.e., the circle of radius one) centered at the origin, such that the arc length counterclockwise along the circle from $(1, 0)$ to Q is β.

Let $k, d \in \mathbb{R}$ with $d \geq 0$. Replacing x by kx and multiplying both sides of the above equation by d yields the equation of the function $g(x)$ obtained by starting with $f(x) = \sin x$, shifting to the left by β and horizontally compressing by a factor of k (i.e., stretching by $1/k$). The resulting general transformation of $\sin x$ is:

$$\boxed{g(x) = d\sin(kx + \beta) = d\,(\cos\beta \sin kx + \sin\beta \cos kx)} \tag{4}$$

Now let us consider an arbitrary function of the form

$$h(x) = A\sin kx + B\cos kx\,, \tag{4}$$

where $A, B \in \mathbb{R}$ are any numbers. The point (A, B) has distance $\sqrt{A^2 + B^2}$ from the origin. If A and B are not both zero, then letting

$$a = \frac{A}{\sqrt{A^2 + B^2}}\,, \quad b = \frac{B}{\sqrt{A^2 + B^2}}\,,$$

the point (a, b) has distance 1 from the origin, hence lies on the unit circle centered at the origin. Thus there is an angle β for which $a = \cos\beta, b = \sin\beta$, and letting $d = \sqrt{A^2 + B^2}$ we have

$$\begin{aligned} h(x) &= d\,(a\sin kx + b\cos kx) \\ &= d\,(\cos\beta \sin kx + \sin\beta \cos kx) \\ &= d\sin(kx + \beta)\,. \end{aligned}$$

Therefore $h(x)$ is a transformation of $\sin x$ having the form (3), where $d = \sqrt{A^2 + B^2}$. The angle β is called the *phase shift*, and the number $d \geq 0$ is the *amplitude*.

Example. Consider the function $h(x) = 3\sin x + 2\cos x$. We have $A = 3$, $B = 2$, $d = \sqrt{3^2 + 2^2} = \sqrt{13}$, $a = \frac{3}{\sqrt{13}}$, and $b = \frac{2}{\sqrt{13}}$. The angle β is an acute angle (since the point $(3, 2)$ lies in the first quadrant, so β can be found on a calculater by taking $\arcsin \frac{2}{\sqrt{13}} \approx 0.588$. Thus we have

$$\begin{aligned} h(x) &= \sqrt{13}\,(\frac{3}{\sqrt{13}}\sin x + \frac{2}{\sqrt{13}}\cos x) \\ &= \sqrt{13}\,(\cos\beta \sin x + \sin\beta \cos x) \\ &= \sqrt{13}\sin(x + \beta)\,, \end{aligned}$$

where $\beta \approx 0.588$. The amplitude is $\sqrt{13}$ and the phase shift is $\beta \approx 0.588$.

Vibrations. We will use the term *vibration* to mean an oscillation having a pattern which repeats every interval of P units of time. The frequency of the vibration, i.e., the number of repetitions of its pattern per unit of time, is $1/P$. For our purposes, time will be measured in seconds. If we realize a vibration as the up and down motion of a point, the vibration is given by a function $y = f(t)$ where y is the position of the particle at time t. The function will be periodic, the period being the number P above.

Vibrating motion can arise from the strings of a violin, a column of air inside a trumpet, of the human vocal chords. The vibration is transmitted through the air by contraction and expansion (This is called a sound wave.) and received by the human ear when the ear drum is set in motion, vibrating in the same pattern as the vibrating object. The brain interprets the vibration as a musical tone. If the vibration has period P, measured in seconds, then the pitch, or frequency, of the tone will be $F = 1/P$ Hz.

Musical Tones and Periodic Functions. Given any periodic function $y = f(t)$ of period P, we can contemplate an oscillating object whose position at time t is $f(t)$ and ask what is the sound of such a vibration. We would expect the pitch of the tone to be $1/P$ Hz, but we wish to investigate what other aspects of $y = f(t)$ determine the character, or timbre, of the sound we are hearing.

If a function $y = f(t)$ did in fact represent the position of an object, we would expect the function $f(t)$ to be continuous. This is based on the supposition that the objects position does not "jump" instantly. Although this is indeed a reflection of reality, our discussion will nevertheless associate a vibration with any periodic function $y = f(t)$ of period $P \in \mathbb{R}^+$ satisfying the following more general properties:

(1) f has only finitely many discontinuities on $[0, P)$.
(2) f is *bounded*, i.e., there are numbers $b, B \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, $b < f(t) < B$.

We interpret the discontinuities as moments at which the vibrating object's position changes very quickly, so that the transition from one location to another seems instantaneous. This exemplifies the fact that mathematics presents a models of physical phenomena, not an exact representation.

Suppose $y = f(t)$ is a periodic function, with period P, satisfying the above two conditions. As described above, $f(t)$ is associated to a tone of pitch (frequency) $F = 1/P$. According to our observations about the effect of shifting on periodicity, the pitch is not changed if we alter $f(t)$ by a horizontal shift. Since such a shift can be thought of as a delay, we would not expect the timbre of the tone, and in fact it does not. The vertical shift describes a motion with altered amplitude, but the same pitch and the same basic "personality". Observation confirms that such a stretch adjusts the loudness, with very little effect, if any, on the timbre of the tone. The horizontal compression $y = f(ct)$ changes the period to P/c, hence the pitch to $1/(P/c) = c/P = cF$. So the effect of compressing horizontally by a factor of c is to multiply the frequency of $f(t)$ by c.

Effect of Horizontal Stretching on Pitch. The final observation above tells us how to apply a horizontal compression to $f(t)$ to achieve any desired pitch (frequency) r. Suppose the period P is given in seconds. We want $r = cF = \frac{c}{P}$, which gives $c = rP$. Thus the function

$$y = f(rPt)$$

represents a tone having frequency r cycles per second, i.e., r Hz.

Example. Suppose $y = \sin x$ gives motion in seconds. Here $P = 2\pi$, so the frequency is

$1/2\pi$ Hz (which is way below the threshold of human audibility). Let us adjust the pitch to give A_4, tuned to $r = 440$ Hz. Accordingly we write $y = \sin(rPt)$, i.e.,

$$y = \sin(880\pi t)\,.$$

The tone given by a sine function as above is sometimes called a "pure tone". It is a nondescript hum, very similar to the tone produced by a tuning fork.

Fourier Theory. We will describe how all periodic functions having reasonably good behavior can be written in terms of the functions $\sin t$ and $\cos t$. We first make the following observations.

The first is that if $f(t)$ and $g(t)$ are two functions which are periodic of period P, then so is $(f+g)(t)$, which is defined as $f(t)+g(t)$. This is elementary: $(f+g)(t+P) = f(t+P)+g(t+P) = f(t)+g(t) = (f+g)(t)$. More generally, one sees that $f_1(t), \ldots, f_n(t)$ are periodic of period P then so is $\sum_{k=1}^{n} f_k(t)$.

Secondly, suppose $f(t)$ is periodic of period P, and $k \in \mathbb{Z}^+$. As we have seen, the function $f(kt)$ has as its graph the graph of $f(t)$ compressed horizontally by a compression factor of k, and it has period P/k. However, it also has period P, since $f(k(t+P)) = f(kt+kP) = f(kt)$. Obviously the function $af(kt)$, for any $a \in \mathbb{R}$, is also periodic of period P. Therefore a sum $\sum_{k=1}^{n} a_k f(kt)$, where $a_1, \ldots, a_n \in \mathbb{R}$, is again periodic of period P. In particular, a sum $\sum_{k=1}^{n} a_k \sin(kt)$ has period 2π.

The following theorem entails two concepts which go well beyond the scope of this course: the derivitive and the infinite summation.

THEOREM. *Suppose $f(t)$ is periodic of period 2π which is bounded and has a bounded continuous derivative at all but finitely many points in $[0, 2\pi)$. Then there is a real number C and sequences of real numbers $A_1, A_2, A_3 \ldots$ and $B_1, B_2, B_3 \ldots$ such that, for all t at which $f(t)$ is continuous we have $f(t)$ represented by the convergent sum*

$$f(t) = C + \sum_{k=1}^{\infty} \left[A_k \sin(kt) + B_k \cos(kt)\right]\,. \tag{5}$$

Note that there is a condition on $f(t)$ beyond the conditions (1) and (2) stated earlier in this chapter. It involves the concept of derivitive, which one learns in calculus. The condition roughly says that, away from finitely many points, the graph of $f(t)$ is smooth and that it doesn't slope up or down too much.

The infinite summation, called the *Fourier series* for f, is based on the notions of limit and convergence, also from calculus. With the proper definitions and development, it becomes possible for an infinite sum to have a limit, i.e., to "add up" (converge) to a number. An example is the sum $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which has 2 as its limit.

This moral of the story presented by the above theorem is that well-behaved periodic functions can be approximated by a series of multiples of the sine and cosine functions functions. There is more to the story, which, again, can be understood by anyone familiar with calculus: The coefficients in formula (1) are uniquely determined by the integrals below.

$$C = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt$$
$$A_k = \frac{1}{\pi}\int_0^{2\pi} \sin(kt)f(t)\,dt$$
$$B_k = \frac{1}{\pi}\int_0^{2\pi} \cos(kt)f(t)\,dt$$

If $g(t)$ is a function of arbitrary period P, then $g(\frac{P}{2\pi}t)$ has period 2π, hence we have

$$g\left(\frac{P}{2\pi}t\right) = C + \sum_{k=1}^{\infty}\left[A_k\sin(kt) + B_k\cos(kt)\right]$$

by the theorem. Recovering $g(t)$ by replacing t by $\frac{2\pi t}{P}$ in the above, we get the Fourier series for an arbitrary function of period P, satisfying the other hypotheses of the theorem:

$$g(t) = C + \sum_{k=1}^{\infty}\left[A_k\sin\frac{2\pi kt}{P} + B_k\cos\frac{2\pi kt}{P}\right] \tag{6}$$

Harmonics and Overtones. Associating the function $g(t)$ having period P as above with a musical tome of pitch $F = 1/P$, let us note that

$$g(t) = C + \sum_{k=1}^{\infty}\left[A_k\sin 2\pi Fkt + B_k\cos 2\pi Fkt\right]$$

each summand $A_k\sin 2\pi Fkt + B_k\cos 2\pi Fkt$ in (6) has the form (4), and therefore represents a transformation of $\sin 2\pi Fkt$ which can be written in the form (3) as

$$d_k\left(\cos\beta_k\sin 2\pi Fkt + \sin\beta_k\cos 2\pi Fkt\right) = d_k\sin(2\pi Fkt + \beta_k)\,,$$

where

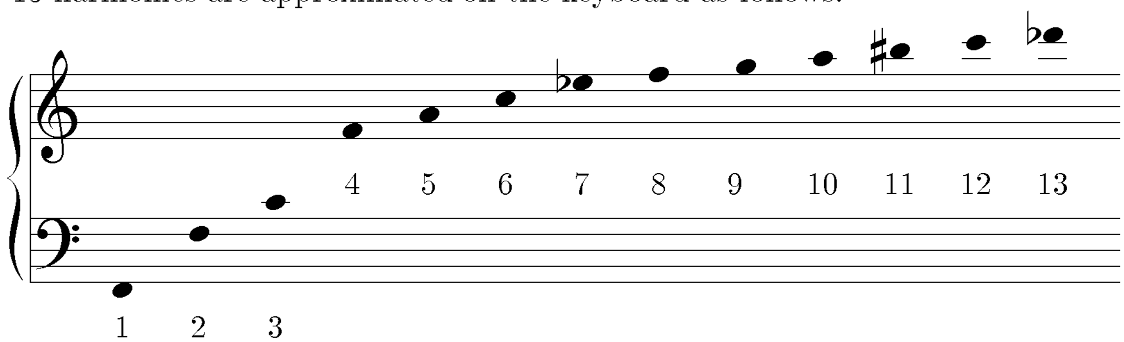
$$d_k = \sqrt{A_k^2 + B_k^2}, \qquad \cos\beta_k = \frac{A_k}{d_k}, \qquad \sin\beta_k = \frac{B_k}{d_k}$$

(provided A_k and B_k are not both zero). Hence we have

$$\boxed{g(t) = C + \sum_{k=1}^{\infty} d_k\sin(2\pi Fkt + \beta_k)}$$

The k^{th} summand $d_k\sin(2\pi Fkt + \beta_k)$ is obtained from $\sin t$ via shifting by β_k (the k^{th} phase shift), compressing by a factor of k and stretching vertically by a factor of d_k (the k^{th}

amplitude). This function has the same basic sound (pitch and timbre) as $\sin(2\pi kFt)$, with a volume adjustment resulting from the amplitude d_k. It is called the k^{th} *harmonic* of the function $g(t)$. For $k \geq 1$ it is also called the $(k-1)^{\text{th}}$ *overtone* of $g(t)$. When isolated, this harmonic gives the pitch kF, so the sequnce of pitches associated to the harmonics gives the sequence of integer ratios with the fundamental frequncy F. These are the intervals discussed in Chapter IX; recall that is we take F_2 as the fundamental (first harmonic), the first 13 harmonics are approximated on the keyboard as follows:

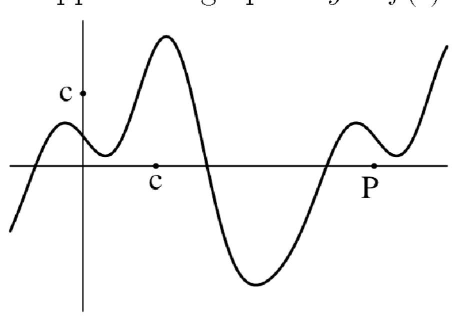


It is the relative sizes of the (non-negative) amplitudes $d_1, d_2, d_3, \ldots$ that determines the timbre, or character of a sustained tone, allowing us to distinguish between different musical voices and instruments. We can think of d_k as the "weight" or "degree of presence" of the k^{th} harmonic in the sound represented by $g(t)$. The timbre of the tone seems to depend on this sequence alone, independent of the sequence of phase shifts $\beta_1, \beta_2, \beta_3, \ldots$, which certainly affect the shape of the graph of $g(t)$, but not the sound.

Exercises

(1) Prove that if $y = f(t)$ has period P, then so does $y = f(t) + c$, $y = f(t - c)$, and $y = cf(t)$, for any $c \in \mathbb{R}$. Prove that $f(t/c)$ $(c \neq 0)$ has period cP.

(2) Suppose the function $y = f(t)$ is the periodic function of period P corresponding to a musical tone, and suppose the graph of $y = f(t)$ is:



For each of the functions below, sketch its graph and explain how its associated tone compares that of $f(t)$.

(a) $y = \frac{1}{2}f(t)$

(b) $y = f(2t)$

(c) $y = f(t) + c$

(d) $y = f(t + c)$

(3) Find the value α for which the pitch associated to the periodic function $y = \sin(\alpha t)$, where t is time in seconds, is:

(a) middle C (b) $A_\flat^2$ (c) $D_\sharp^6$

(4) Find the period, frequency, amplitude, and phase shift for these functions, and express each in the form $A\sin(\alpha t) + B\cos(\alpha t)$:

(a) $f(t) = 5\sin(30\pi t + \frac{\pi}{4})$

(b) $g(t) = \sqrt{2}\sin(800t + \pi)$

(c) $h(t) = -\frac{5}{3}\sin(2000t + \arcsin(0.7))$

(5) Find the period, frequency, amplitude, and phase shift for these functions, and express each in the form $d\sin(\alpha t + \beta)$:

(a) $f(t) = 4\sin(300t) + 5\cos(300t)$

(b) $g(t) = 2\sin(450\pi t) - 2\cos(450\pi t)$

(c) $h(t) = -\sin(1500\pi t) + 3\cos(1500\pi t)$

(6) Suppose musical tone with pitch B_4 has harmonics 1, 3, 5 only, with amplitudes 1, $\frac{1}{9}$, $\frac{1}{25}$, respectively, and phase shifts 0, π, $-\frac{\pi}{2}$, respectively. Suppose also that the vertical shift C is 0. Write its Fourier series in the form $\sum A_k\sin(kt) + B_k\cos(kt)$.