CHAPTER XI

THE RATIONAL NUMBERS AS INTERVALS

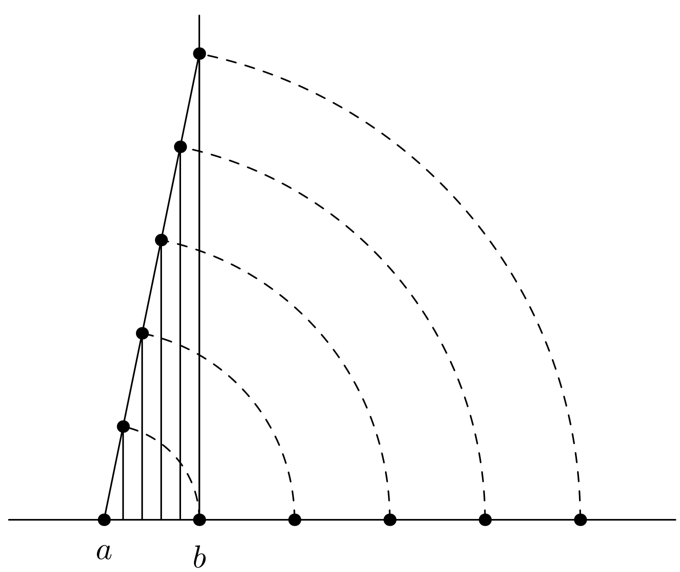
The rational numbers $\mathbb{Q}$ are a subset of $\mathbb{R}$ containing $\mathbb{Z}$, and we also have the containments $\mathbb{Z}^+ \subset \mathbb{Q}^+ \subset \mathbb{R}^+$. We have noted that elements of $\mathbb{R}^+$ are in 1-1 correspondence with the set of intervals, and that this gives a group isomorphism from the group of intervals with $(\mathbb{R}^+, \cdot)$. In the last chapter we examined those intervals which correspond to positive integers, i.e., lie in the monoid $(\mathbb{Z}^+, \cdot)$. Now we will consider those intervals which correspond to elements of the subgroup $(\mathbb{Q}^+, \cdot)$.

It has long been acknowledged that two pitches sounded simultaneously create an effect that we are prone to call "harmonious" or "consonant" when the ratio of their frequencies can be expressed as a ratio $(n : m)$ where m and n are small positive integers. The smaller the integers, the more consonant the interval. We refer to such intervals as *just* intervals.

Rational Intervals. To say that an interval I is given by a ratio $(n : m)$ of positive integers is to say that it corresponds to a positive rational number.

Definition. An interval I will be called *rational* if its corresponding ratio lies in $\mathbb{Q}^+$. Otherwise we say I is an *irrational* interval.

In ancient times such intervals could be accurately created with a vibrating string of length L using techiques of plane geometry. Any interval can be divided into n equal subintervals using compass and rule, as shown below for $n = 5$ and the interval $[a, b]$.



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If $\frac{n}{m} \geq 1$, its rational interval is obtained with the strings fundamental frequency by fretting the string at distance $\frac{m}{n} \cdot L$ ($\leq L$) from one end.

The intervals of equal temperament, however, were not so accessible before the development of relatively modern techniques. For example the semitone's ratio of $2^{1/12}$ would necessitate, as we have seen, finding the distance $2^{-1/12}L$ - a technique not accessible to ancient mathematicians.

Unique Factorization of Positive Rational Numbers. The following theorem about factorization in $\mathbb{Q}^+$ follows from the analogous theorem about $\mathbb{Z}^+$ from Chapter VII.

THEOREM. *Let $x \in \mathbb{Q}^+$. Then x can be factored as*

$$x = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

where $r \geq 0$, $p_1, p_2 \ldots, p_r$ are distinct primes, and $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{Z}, \neq 0$. Moreover, this factorization is unique, meaning that if $x = q_1^{\beta_1} q_2^{\beta_2} \cdots q_t^{\beta_t}$ is another such factorization, then $t = r$ and after rearranging we have $p_1 = q_1, p_2 = q_2, \ldots, p_r = q_r$.

Note that this statement differs from the analogous theorem about $\mathbb{Z}$ in that it allows the exponents $\alpha_1, \ldots, \alpha_r$ to be <u>non-zero</u> integers, not just positive ones. The proof of this theorem will be an exercise.

Given $x = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ as in the theorem, we may, without loss of generality, assume that $\alpha_1, \ldots, \alpha_i$ are positive and $\alpha_{i+1}, \ldots, \alpha_r$ are negative. Set $s = r - i$, and let $\beta_j = -\alpha_{i+j}$ and $q_j = p_{i+j}$ for $j = 1, \ldots, s$. We have

$$x = \frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i}}{q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}}$$

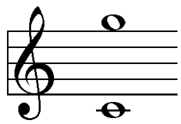
with $p_1, \ldots, p_i, q_1, \ldots, q_s$ being distinct primes and $\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_s$ <u>positive</u> integers. We can easily write an element of $\mathbb{Q}$ in this form provided we can find the prime factorization of its numerator and denominator. The fraction $x = \frac{1{,}222{,}452}{11{,}180{,}400}$ seems intractible, but a little work with small primes gives the factorizations $1{,}222{,}452 = 11 \cdot 7^3 \cdot 3^4 \cdot 2^2$, $11{,}180{,}400 = 11^3 \cdot 7 \cdot 5^2 \cdot 3 \cdot 2^4$. Thus, by cancellation, we have

$$x = \frac{7^2 \cdot 3^3}{11^2 \cdot 5^2 \cdot 2^2}\,.$$

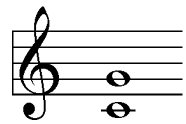
We will seek to understand rational intervals by the configuration of prime numbers $p_1, \ldots, p_i, q_1, \ldots, q_s$ which appear in their factorization, as above. We will first focus on some just intervals which are less than an octave, comparing them to their keyboard approximations.

We begin considering some cases where the denominator is a power of 2, i.e. rational intervals having ratio $n/2^\beta$, where n is odd. In this case, the interval is the composition of the integral interval n with $-\beta$ octaves. For example:

Just Fifth. Consider the interval given by $\frac{3}{2} \in \mathbb{Q}^+$. This is the integral interval 3 lowered by 1 octave. We noted in Chapter 8 that the interval 3 is ≈ 1.96 cents sharp of the keyboard's octave plus a fifth. Hence $\frac{3}{2}$ is sharp of a fifth by this same amount. (Or we can calculate directly: $1200 \log_2 \frac{3}{2} \approx 701.96$. The keyboards fifth is 700 cents.) The rational interval given by $\frac{3}{2}$ is called the *just fifth.*

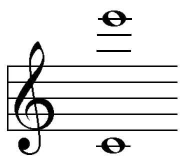


approximation of 3 ($\approx$ 2 cents flat)



approximation of $\frac{3}{2}$ ($\approx$ 2 cents flat)

Just Major Third. The interval $\frac{5}{4}$ is the integral interval 5 minus two octaves, which is about 14 cents less than the keyboard's two octaves plus a major third. Hence $\frac{5}{4}$ is the same amount flat of the major third, and is called the *just major third.*



approximation of 5 ($\approx$ 14 cents sharp)

approximation of $\frac{5}{4}$ ($\approx$ 14 cents sharp)

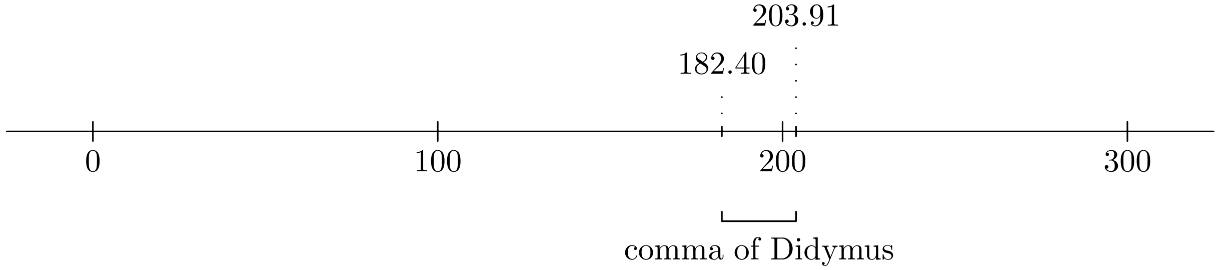
Greater Whole Tone (Pytharean Whole Tone). Since 3 is approximately one octave plus a fifth, the interval $\frac{9}{8}$ is twice that, lowered by three octaves: $\frac{9}{8} = (\frac{3}{2})^2 \cdot \frac{1}{2}$. This gives something close to the keyboard's step. The calculation $1200 \log_2 \frac{9}{8} \approx 203.91$ shows that this just interval is about 4 cents sharp of a step. We refrain from calling this interval the "just step" or "just whole tone" because we will soon encounter another just interval that is well approximated by the keyboard's one step. Instead, we will refer to this interval as the *greater whole tone.* It is also called the *Pythagorean whole tone,* for a reason that will be given shortly.

Now we will investigate some intervals having ratio $n/3^\beta$, where n is not divisible by 3.

Just Fourth. The most basic of these is the interval given by the ratio $\frac{4}{3}$. Note that this interval, call it I, is complimentary to the just fifth, since $\frac{4}{3} \cdot \frac{3}{2} = 2$. This says I is given additively as one octave minus a just fifth, which means it is about 2 cents flat of a fourth. We call I the *just fourth.*

approximation of $\frac{4}{3}$ ($\approx$ 2 cents sharp)

Lesser Whole Tone. The ratio $\frac{10}{9}$ gives another interval approximated by the step. We have $1200 \log_2 \frac{10}{9} \approx 182.40$, showing this interval to be about 18 cents flat of the keyboard's step. This interval will be called the *lesser just step.* Observe that the keyboard's step lies between the lesser and greater just steps, closer to the latter, as indicated on the scale of cents below:



The interval between the lesser and greater whole tones has a ratio of $\frac{9}{8} \div \frac{10}{9} = \frac{81}{80}$ which is measured in cents by $1200 \log_2 \frac{81}{80} \approx 21.50$. This is called the *comma of Didymus.*

Just Major Sixth. The fraction $\frac{5}{3}$ gives us an interval ≈ 884.36 measured in cents. Since it is only about 16 cents flat of the major sixth, we call it the *just major sixth.*

approximation of $\frac{5}{3}$ ($\approx$ 16 cents sharp)

There are at least two common just intervals whose denominators involve the prime number 5.

Just Minor Third. By virtue of the equality $\frac{3}{2} \div \frac{5}{4} = \frac{6}{5}$, the ratio $\frac{6}{5}$ gives the interval I "between" the just third and the just fifth, i.e., I is (additively) a just fifth minus a just third. The cents measurement of $\frac{6}{5}$ is $1200 \log_2 \frac{6}{5} \approx 315.64$, about 16 cents sharp of the keyboard's minor third. We call it the *just minor third.*

approximation of $\frac{6}{5}$ ($\approx$ 16 cents flat)

Just Semitone. We consider the fraction $\frac{16}{15} = \frac{2^3}{3 \cdot 5}$. By virtue of the fact that it has larger numerator and denominator than any of those previously discussed, it gives an interval that might be considered "less just", and which one might expect to be less consonant. It is also the first ratio we have listed whose denominator involves more than one prime. We have $1200 \log_2 \frac{16}{15} \approx 111.73$, placing this interval about 12 cents sharp of one semitone. It is called the *just semitone.*

approximation of $\frac{16}{15}$ ($\approx$ 12 cents flat)

Septimal Intervals. All the just intervals above involve only the primes 2,3, and 5. (According to a definition that will be given in the next chapter, these are *5-limit intervals.*) The prime 7 introduces us to still more just intervals, and these intervals are not so well approximated by 12-tone equal temperament. Three examples are the *septimal minor seventh,* with ratio $\frac{7}{4}$, its compliment the *septimal whole tone,* with ratio $\frac{8}{7}$, and the *septimal minor third,* with ratio $\frac{7}{6}$. These intervals are sufficiently far away from keyboard notes as to impart a texture that is sometimes called "blue", or "soulful", to music which employs them.

approximation of $\frac{7}{4}$ ($\approx$ 31 cents sharp)

approximation of $\frac{8}{7}$ ($\approx$ 31 cents flat)

approximation of $\frac{7}{6}$ ($\approx$ 33 cents sharp)

Higher Primes. There is, in theory, an infinite list of higher primes to consider (The infinitude of the set of primes will appear as an exercise.), but in practice there is a limit to the audible range of an interval. The human ear is able to listen to ratios up to about 1000, so there are indeed many audible possibilities. Although the primes > 7 may be considered remote, they can contribute intervals which have legitimate uses in music. These intervals, unnamed for the most part, will be referred to as "exotic". We often describe them by using the prefixes sub- and super- before the names of keyboard intervals.

An example is the exotic tritone with ratio $\frac{11}{8}$. The calculation $1200 \log_2 \frac{11}{8} \approx 551.32$ shows this interval to be almost halfway between the keyboard's fourth and tritone. Another is the exotic super-minor sixth given by $\frac{13}{8}$. It is about 41 cents sharp of the keyboard's minor sixth, according to $1200 \log_2 \frac{13}{8} \approx 840.53$. These intervals are quite strange to the ear.

By contrast, the next two primes 17 and 19 give near-keyboard intervals as follows: A super-semitone is given by $\frac{17}{16}$, which is only about 5 cents sharp of the keyboard's semitone, and a sub-minor third is given by $\frac{19}{16}$, only 2 cents flat of the keyboard's minor third. Note that these rational intervals are better approximated by the keyboard than the just semitone and the just minor third.

The Comma of Pythagoras. The Greek mathematician Pythagoras (c. 540-510 B.C.) believed that the perfection of the $(3:2)$ fifth (what we now call the just fifth) symbolized the perfection of the universe. He discovered that the iteration of twelve just fifths is almost the same as the iteration of seven octaves. This is demonstrated by:

$$\left(\frac{3}{2}\right)^{12} = \frac{3^{12}}{2^{12}} = \frac{531441}{4096} \approx 129.75$$

$$2^7 = 128$$

The interval between these is

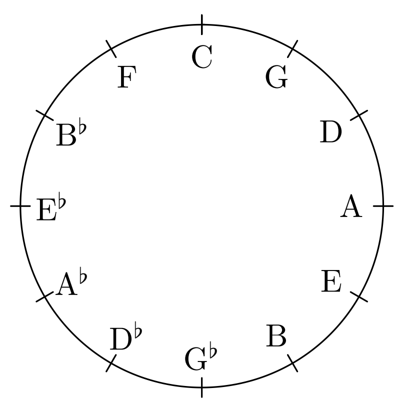
$$\frac{\left(\frac{3}{2}\right)^{12}}{2^7} = \frac{3^{12}}{2^{19}} = \frac{531441}{524228} \approx 1.01364\,,$$

which is measured in cents by

$$1200\,\log_2\left(\frac{3^{12}}{2^{19}}\right) \approx 23.46\,.$$

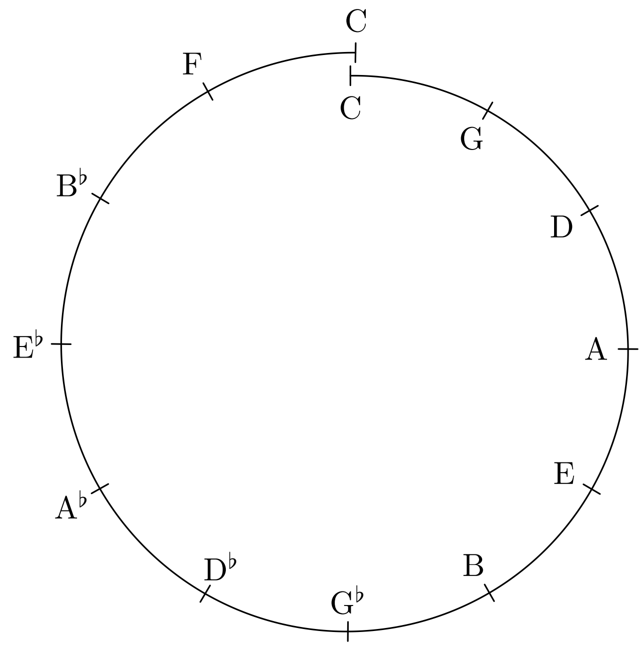
It is called the *comma of Pythagoras.*

This comma is the discrepancy we would get if we were to try to tune up the 12-note scale with just fifths. We know that the tempered fifth, iterated twelve times, gives us seven octaves. Thus if we let the circle represent all intervals (additively) modulo 7 octaves, then a fifth will be $1/12^{th}$ of the circle. Placing C at the top, we have:



12-octave clock with tempered fifths

If we plot the 7-octave circle of fifths using just fifths, the twelve intervals add up to slightly more than one rotation, wrapping around the clock and ending up clockwise of the 12 o'clock position by precisely the comma of Pythagoras, as shown below. Thus the tuning of C would be problematic.



7-octave clock with just fifths

This small but nonnegligible interval, not quite a quarter of a semitone, was greatly disturbing to Pythagoras.

Irrationality of Equally-Tempered Intervals. We will now show that all intervals between notes of the equally tempered scale, excepting iterations of the octave, are irrational, that is, they correspond to ratios which lie outside of $\mathbb{Q}$. In fact, this holds even in non-standard equally tempered scales.

THEOREM. *Let I be an the interval between two notes in the chromatic n-scale. If I is not the iteration of octaves (i.e., the ratio corresponding to I is not a power of 2), then I is an irrational interval.*

PROOF. Suppose I has ratio $x = \frac{a}{b}$, with $a, b \in \mathbb{Z}^+$. By cancelling, we may assume that $\gcd(a, b) = 1$. The modular chromatic interval given by I, which lies in $\mathbb{Z}_{12}$, has finite order, which says that n iterations of I gives k octaves, for some positive integer n and some integer k. This says that $x^n = \left(\frac{a}{b}\right)^n = 2^k$. Writing $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $b = q_1^{\beta_1} \cdots q_t^{\beta_t}$, with $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ being positive integers, as in the Unique Factorization Theorem for $\mathbb{Z}$, we have $\{\{p_1, \ldots, p_r\} \cap \{q_1, \ldots, q_s\} = \emptyset$ (since $\gcd(a, b) = 1$) and

$$x^n = \left(\frac{a}{b}\right)^n = \frac{p_1^{n\alpha_1} \cdots p_r^{n\alpha_r}}{q_1^{n\beta_1} \cdots q_t^{n\beta_t}} = 2^k\,.$$

The uniqueness of this expresssion says that 2 is the only prime in the set of primes $\{p_1, \ldots, p_r, q_1, \ldots, q_s\}$. Therefore either $r = 0$ and $b = 2^\beta$, or $s = 0$ and $a = 2^\alpha$. Hence x has the form 2^α or $1/2^\beta$, which says I is an iteration of octaves. □

This tells us that none of the intervals in any equally-tempered scale, save multi-octaves, are just, and suggests that none should be considered perfectly consonant. However, we have seen that the just intervals involving small powers of the primes 2,3, and 5 have fairly close approximations in the 12-chromatic scale. For example, we have seen that the tempered fifth closely approximates $\frac{3}{2}$, and the tempered major third gives a fair approximation of $\frac{5}{4}$. This likely explains why the 12-chromatic scale gained acceptance. In the next chapter we will present some historical alternate methods of tuning the scale in non-equal temperament designed to render some just intervals with precision, and discuss the advantages and limitations of such scales.

Exercises

(1) For each of these rational intervals, find the 12-chromatic interval which best approximates it, and calculate the error. Express the approximating interval by name (e.g., "minor third").

(a) $\frac{5}{3}$ (b) $\frac{11}{10}$ (c) $\frac{19}{16}$ (d) $\frac{9}{7}$ (e) $\frac{5}{5}$

(2) Give the prime factorizations of these rational numbers as $\frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}}{q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}}$ with

$$\{p_1, \ldots, p_r\} \cap \{q_1, \ldots, q_s\} = \emptyset,$$

writing the primes of the numerator and denominator in ascending order, as in $\frac{2^3 \cdot 5 \cdot 7^2}{3 \cdot 11^2 \cdot 13^3}$.

(a) $\frac{150}{65}$ (b) $\frac{1000}{287}$ (c) $\frac{750}{980}$ (d) $\frac{512}{162}$ (e) $\frac{69}{289}$

(3) Suppose $x \in \mathbb{Q}^+$ has the factorization $x = \frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}}{q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}}$ as in the previous exercise. What criteria about this factorization says x is an integer?

(4) Give a direct proof that $\sqrt{2}$ is irrational. Interpret this as a statement about a musical interval.

(5) Let p be a fixed prime. Verify that set of rational numbers x whose prime factorization has the form $x = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with $p_1, \ldots, p_r \leq p$ forms a subgroup of $(\mathbb{Q}^+, \cdot)$. (Intonation which utilizes only interval ratios in this subgroup is called *p-limit tuning.*)

(6) Show by multiplication and division in $\mathbb{Q}^+$ that:

(a) A just major third plus a just minor third is a just fifth.
(b) A just fifth plus a septimal minor third is a just minor seventh.
(c) A greater just whole tone plus a lesser just whole tone is a just major third.
(d) The comma of Didymus plus one octave is two just fifths minus a lesser whole tone.
(e) A just major third minus a just fourth is a just semitone downward.

(7) Show by comparing rational numbers that:

(a) Three just major thirds is not an octave.
(b) Four just minor thirds is not an octave.
(c) A just fifth plus two just semitones is not a just major sixth.
(d) Two just fourths is not a just minor seventh.
(e) The difference between a just major third and a just minor third is not a just semitone.

In each case above, calculate the difference as a fractional interval ratio, with prime factorization, and calculate the difference in cents.

(8) Prove that the (additive) measurement in octaves of a rational interval cannot be a rational number unless the interval is a multioctave. (Hint: If interval ratio x is measured by $\frac{a}{b}$ octaves, we have $x = 2^{\frac{a}{b}}$. Use unique factorization in $\mathbb{Q}^+$.) Deduce that the measurement of such an interval in semitones or cents is not rational.