CHAPTER XII

TUNING THE SCALE TO OBTAIN RATIONAL INTERVALS

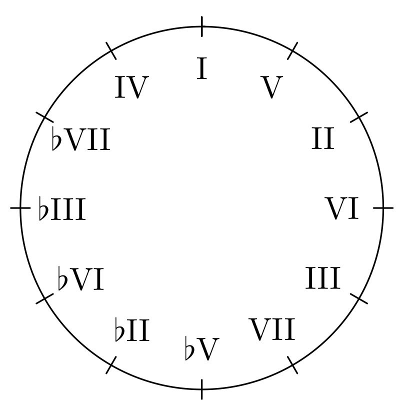
We will now present some other traditional ways to tune the diatonic and chromatic scales in order to render certain intervals as just intervals. An understanding of the advantages and disadvantages of such scales will help to explain why the system of equal temperament eventually gained wide acceptance.

p-Limit Tuning. Given a prime integer p the subset of $\mathbb{Q}^+$ consisting of those rational numbers x whose prime factorization has the form $x = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with $p_1, \dots, p_r \le p$ forms a subgroup of $(\mathbb{Q}^+, \cdot)$. (This will be an exercise.) We say that a scale or system of tuning uses *p-limit tuning* if all interval ratios between pitches lie in this subgroup.

The Pythagorean Scale. This scale, deriving its name from Pythagoras' high regard for the just fifth (ratio 3 : 2), tunes the scale so that all intervals between scale tones are rational intervals involving only the primes 2 and 3. This means it has it has 3-limit tuning: all intervals between scale tones have ratios that can be expressed as $2^\alpha \cdot 3^\beta$. The Pythagorean scale arises from tuning each of the intervals in the upward sequence of scale tones

$$\hat{4} \to \hat{1} \to \hat{5} \to \hat{2} \to \hat{6} \to \hat{3} \to \hat{7}$$

to be 3 : 2. Note that the diatonic root notes occupy seven of the twelve positions on the circle of fifths.



The Pythagorean scale simply tunes each of these fifths between diatonic notes to be just fifths.

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We calculate the interval ratio from $\hat{1}$ to each scale tone within an octave by dividing. For example, the iteration $\hat{1} \to \hat{5} \to \hat{2}$ of two just fifths gives the interval $\frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4}$. Since $2 < \frac{9}{4} < 4$, this interval lies between one and two octaves. So to get the scale tone $\hat{2}$ which is within one octave we divide by 2 to get $\frac{9}{4} \cdot \frac{1}{2} = \frac{9}{8}$. We recognize this interval as the Pythagorean whole tone (greater whole tone), whence the name.

In similar fashion, we calculate the interval between adjacent scale tones $\hat{1}$ and $\hat{3}$ to be

$$\left(\frac{3}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{3^4}{2^6} = \frac{81}{64}$$

This interval, measured in cents is $1200 \log_2 \frac{81}{64} \approx 407.88$, about 8 cents sharp of the tempered major third, and about 22 cents sharp of the just major third.

The ratio of each of the scale tones to the scale tone $\hat{1}$ in the Pythagorean scale is given by this table:

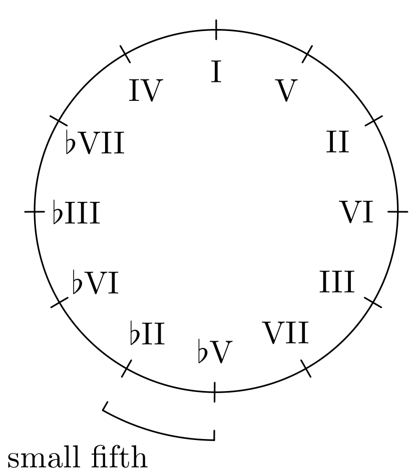
scale tone :	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$	$\hat{7}$	$\hat{8}$
ratio to $\hat{1}$:	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	$\frac{2}{1}$

Pythagorean diatonic scale

In this scale each of the five whole step intervals $\hat{1} \to \hat{2}$, $\hat{2} \to \hat{3}$, $\hat{4} \to \hat{5}$, $\hat{5} \to \hat{6}$, and $\hat{6} \to \hat{7}$ is two just fifths minus an octave, which is the Pythagorean whole tone (greater whole tone). Both of the half step intervals are given by the complicated ratio $\frac{256}{243} = \frac{2^8}{3^5}$, which Pythagoras called a *hemitone.* (The comparison of this interval with the tempered semitone and half the Pythagorean whole tone will appear as an exercise.) Thus the intervals between adjacent scale tones in the Pythagorean scale are given by:

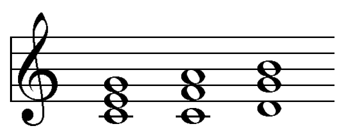
$$\hat{1} \xrightarrow{9:8} \hat{2} \xrightarrow{9:8} \hat{3} \xrightarrow{256:243} \hat{4} \xrightarrow{9:8} \hat{5} \xrightarrow{9:8} \hat{6} \xrightarrow{9:8} \hat{7} \xrightarrow{256:243} \hat{8}$$

The Pythagorean scale can be extended to a chromatic scale by continuing to tune just fifths around the circle of fifths, but, as we have seen, the comma of Pythagoras prevents us from completing the circle using only just fifths. The comma is accommodated by allowing a "small" fifth between some two adjacent positions in the circle. This is often placed at one of the bottom clock positions, either between $\hat{7}$ and $\flat\hat{5}$ or between $\flat\hat{5}$ and $\flat\hat{2}$. Choosing the latter, we get:



A significant weakness of this scale is it's poor representation of the major third as $\frac{81}{64}$, two Pythagorean whole tones. This ratio is even greater than the tempered major third, and is sharp of the just major third by precisely the comma of Didymus (exercise). We will call it the *Pythagorean major third.* The sharpness of this interval is easily perceived, and the dissonance heard in the in the major triads I, IV, and V when played in Pythagorean tuning render this system unacceptable for music in which a high level of consonance is desired.

The Just Intonation Scale. This scale employs 5-limit tuning in such a way that the diatonic major triads I, IV, and V are justly tuned, meaning that when these chords are voiced within an octave in root position, the ratios of root, third, and fifth are $4:5:6$. In the key of C, this means the C, F, and G major triads will enjoy the consonance of just intonation.



C F G

This is enough to define a tuning for each diatonic scale note, since each diatonic note is used in at least one of these three chords. The justness of the tonic triad I dictates $\hat{1} \to \hat{3}$ gives $\frac{5}{4}$ and $\hat{1} \to \hat{5}$ gives $\frac{3}{2}$. The justness of V says that $\hat{5} \to \hat{7}$ gives $\frac{5}{4}$, hence $\hat{1} \to \hat{7}$ is $\frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8}$, and that $\hat{5} \to \hat{2}$ is $\frac{3}{4}$ (being downward a just fourth), hence $\hat{1} \to \hat{2}$ is $\frac{3}{2} \cdot \frac{3}{4} = \frac{9}{8}$, which is the greater whole tone. To effect the justness of IV, we deduce in similar fashion that the ratios of $\hat{4}$ and $\hat{6}$ to $\hat{1}$ must be $\frac{4}{3}$ and $\frac{5}{3}$, respectively. Thus the ratios of scale tones to $\hat{1}$ in the just intonation scale are given by:

scale tone :	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$	$\hat{7}$	$\hat{8}$
ratio to $\hat{1}$:	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$

Just intonation diatonic scale

Note the more consonant intervals with $\hat{1}$ given by scale notes $\hat{3}$, $\hat{6}$, and $\hat{7}$ over those of the Pythagorean scale.

In the just intonation scale the greater whole tone appears as the interval $\hat{1} \to \hat{2}$, $\hat{4} \to \hat{5}$, and $\hat{6} \to \hat{7}$, while $\hat{2} \to \hat{3}$ and $\hat{5} \to \hat{6}$ are the lesser whole tone. Both $\hat{3} \to \hat{4}$ and $\hat{7} \to \hat{8}$ are the just semitone. The intervals between adjacent scale tones in the just intonation scale are as follows:

$$\hat{1} \xrightarrow{9:8} \hat{2} \xrightarrow{10:9} \hat{3} \xrightarrow{16:15} \hat{4} \xrightarrow{9:8} \hat{5} \xrightarrow{10:9} \hat{6} \xrightarrow{9:8} \hat{7} \xrightarrow{16:15} \hat{8}$$

In addition to giving justly tuned major triads I, IV, and V, the just intonation scale gives justly tuned minor triads IIIm and VIm, and justly tuned minor sevenths IIIm^7 and VIm^7. (Recall that the minor triad in root position, voiced within an octave, is justly tuned as $10:12:15$ and the minor seventh is justly tuned as $10:12:15:18$.)

The just intonation scale is extended to a chromatic scale in such a way as to render certain other triads in just intonation, as follows.

(1) $\flat\hat{6}$ and $\flat\hat{3}$ are tuned so that $\flat$VI is justly tuned. This places $\flat\hat{6}$ a just major third below $\flat\hat{8}$ and $\flat\hat{3}$ a just minor third above $\hat{1}$.
(2) $\flat\hat{7}$ is tuned a just minor third above $\hat{5}$. This makes $\flat$III a justly tuned major triad.
(3) $\flat\hat{2} = \sharp\hat{1}$ is tuned so that VI is justly tuned.
(4) $\flat\hat{5} = \sharp\hat{4}$ is tuned to be a just fourth below $\hat{7}$. This makes VIIm a justly tuned minor triad.

With this chromatic scale many, but not all, of the major triads, minor triads, and minor sevenths are justly tuned. For example, Im^7 and IVm^7 are just, but the major triads II and III are both bad, the former having a flat fifth and the latter having a sharp third. This, unfortunately, precludes have well-tuned chords any extensive circle-of-fifths root movement.

Since the just intonation scale utilizes 5-limit tuning, there are no just sevenths. Several of the seventh chords, such as I^7, have the tuning obtained by placing the seventh a just minor third (ratio $\frac{6}{5}$) above the fifth. This gives a seventh tuned as $20 : 25 : 30 : 36$, which is decidedly less consonant than the just seventh $4 : 5 : 6 : 7$. Even worse, the most needed seventh in conventional harmony, V^7, is even less consonant, since the interval $\hat{2} \to \hat{4}$ is not a just minor third. An easy exercise in arithmetic shows this chord has the tuning $36 : 45 : 54 : 64$.

The Classical Mean-Tone Scale. The acceptance of thirds into Western music, which occurred in the 14th and 15th centuries, brought the need for tuning which goes beyond the 3-limit. Certain compromises were introduced which detuned fifths in order to improve the sound of thirds. Such scales are called *mean-tone* scales. We will discuss the one which became most common, called the *classical mean-tone scale;* henceforth this is what we will mean when we use the term "mean-tone scale".

Unlike the Pythagorean and just intonation scales, the mean-tone scale allows some irrational intervals. All of its rational intervals lie in the subgroup of $\mathbb{Q}^+$ consisting of those elements whose prime factorization involves only 2 and 5.

The idea of the mean-tone scale is to shrink the fifths around the clock equally so that the major third spanning four clock positions (modulo octave) is the just major third, having ratio $5 : 4$. One way to calculate the ratio r of such a fifth is to note that 4 iterations of this interval should equal 2 octaves plus a just third, i.e.,

$$x^4 = 4 \cdot \frac{5}{4} = 5$$
$$\text{therefore } x = \sqrt[4]{5} = 5^{\frac{1}{4}} \approx 1.49535\,.$$

(Note that the closeness of this ratio, ≈ 1.49535, to $1.5 = \frac{3}{2}$.) This is an irrational interval (exercise) whose measurement in cents is calculated by

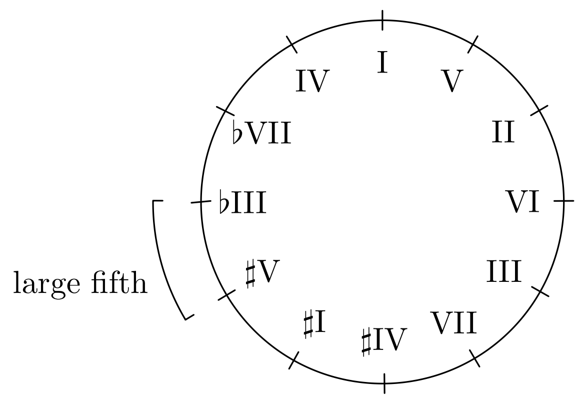
$$1200\ \log_2 \sqrt[4]{5} = 1200\ \log_2 5^{\frac{1}{4}} = \frac{1200}{4} \log_2 5 = 300\ \log_2 5 \approx 696.58\,.$$

Thus the *mean-tone fifth* lies about 3 cents flat of the tempered fifth and about 5 cents flat of the just fifth – tolerably close.

However, as with the Pythagorean scale, there must be a comma in the circle. The problem is that three just thirds do not constitute an octave, as seen by:

$$\left(\frac{5}{4}\right)^3 = \frac{125}{64} < \frac{128}{64} = 2$$

Hence if we tune fifths around the clock so that every four consecutive clock positions equals (modulo cotave) a just third, then the twelfth position does not coincide with the starting point, being flat by the interval ratio $2/(\frac{125}{64}) = \frac{128}{125}$, which is about 41 cents. Therefore a "large fifth" is placed somewhere on the lower left side of the clock, usually located so that it does not occur between diatonic scale tones. For example, it could be placed between the 8 and 9 o'clock positions, as depicted below.



The large fifth which appears in the mean-tone chromatic scale, lying about two-fifths sharp of a semitone sharp of the just fifth or tempered fifth, was long ago dubbed the *wolf fifth,* after the animal's howl.

With the wolf-fifth placed away from the diatonic clock positions, each diatonic whole-tone equals two mean-tone fifths minus an octave, which has ratio $(\sqrt[4]{5})^2/2 = \sqrt{5}/2$, calculated in cents by $1200 \log_2 \frac{\sqrt{5}}{2} \approx 193.157$, about 7 cents flat of the tempered step.

We can calculate the interval between I and each of the scale tones. For example, the interval between scale tones I and VI is three mean-tone fifths minus an octave, has ratio $5^{3/4}/2$. Here is a table of such ratios.

scale tone :	I	II	III	IV	V	VI	VII	VIII
ratio to I :	$\frac{1}{1}$	$\frac{\sqrt{5}}{2}$	$\frac{5}{4}$	$\frac{2}{5^{\frac{1}{4}}}$	$\frac{5^{\frac{1}{4}}}{1}$	$\frac{5^{\frac{3}{4}}}{2}$	$\frac{5^{\frac{5}{4}}}{4}$	$\frac{2}{1}$

mean-tone diatonic scale

Problems with Unequal Temperament. Each of the three scales discussed in the chapter has certain advantages. The Pythagorean scale give a just fifth almost always. The Just intonation scale gives many perfectly tuned chords. The mean-tone scale, with the comma placed as above, gives tolerably tuned major triads on each of the diatonic roots except VII, and even this triad would be well-tuned if we rotated the comma one position clockwise. (Why?)

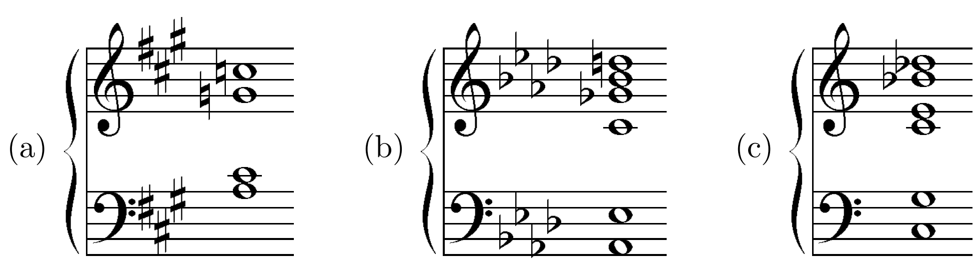
One obvious drawback of each of these systems is that some intervals are poorly tuned. This Pythagorean scale gives poor thirds, and one weak fifth. The just intonation scale renders the major triads II and III with less-than-desirable tuning. The meantone scale has one very bad fifth.

But a more serious problem with these scales is their asymmery with respect to the choice of key. A keyboard instrument cannot be conveniently retuned between songs, or in the middle of a piece that changes key. If the keyboard is tuned to the just intonation scale in the key of C, a song in D has serious problems because the tonic triad is distractingly out of tune. This is what fed the gradual adoption of equal temperament. In the equally tempered scale, one has to accept sharp thirds and sevenths, but all chords of the same type are tuned precisely the same, regardless of their root relative to the key. While this imperfect tuning was a bitter pill to swallow, it allowed composers and performers to use extensive harmonic variety and freely modulate from one key to another. A great deal of 19th and 20th century music is deeply entrenched in this this liberation.

In the first half of the 18th century J. S. Bach produced a bold demonstration of equal temperament[1] by producing his famous *Well-Tempered Clavier,* a collection of 48 preludes and fugues, assembled in two parts, each part containing 24 pieces representing each of the major and minor keys. Another example of a composition which exploits this harmonic emancipation is Franz Listz's classic piano piece *Liebestraum* (19th century) which extensively traverses the circle of fifths, modulates several times, and uses every root note in the chromatic scale of its initial (and final) key, $A^\flat$.

Exercises

(1) Give just tunings for each of these jazz chords by utilizing "exotic" primes , i.e., primes which are ≥ 7.



(2) (a) The Pythagorean scale's minor third is one greater whole tone plus one hemitone, called a *Pythagorean minor third.* Express it as a ratio and compare it, as ratios and in cents, to the tempered minor third and the just minor third.

(b) Show that the interval between the just major third and the Pythagorean major

[1] Actually, music historians disagree as to whether Bach was actually touting equal temperamant or some other system quite close to equal temperament.

third is the comma of Didymus. Explain why the mean-tone fifth is flat of the just by one-fourth of the comma of Didymus, and use this to recalculate the mean-tone fifth in cents.

(3) Suppose the following passage is tuned so that if a note is a fifth or minor third, modulo octave, from a note in the previous chord, then that interval is just. Show that the final G will be sharp of the initial G by the comma of Didymus.

(This example was presented in 1585 by G. B. Benedetti.)

(4) Which major triads in the mean tone scale have relatively good fifths but poor thirds? (Place the comma between the 8 and 9 o'clock positions.) How does mean tone temperament render the minor third, compared with the just third?

(5) Suppose these chords are in root position, voiced within an octave using the just intonation scale. Give the the reduced ratio for each chord. Which of these chords will be "pleasing" when this is played in the just intonation scale? Define *pleasing* to mean it uses only integers ≤ 20 when expressed as a reduced ratio.

(a) VIm (b) III (c) IV^{+9} (d) $\flat\mathrm{II}^7$ (e) V^6

(6) (a) Identify those fifths which in the just intonation chromatic scale's circle of fifths which are not just fifths $(\frac{3}{2})$, and express each of these "imperfect" fifths as a reduced ratio of integers $n_1 : n_2$.

(b) Assume we have tuned the Pythagorean chromatic scale with the comma placed between VII and $\flat$V. Certain of the major triads with perfect fifths have better thirds by virtue of the fact that the third lies across the comma from the root and fifth. Calculate the difference of this third from the just third in cents and identify by Roman numeral which major triads possess this property.