EXAM II

Math 109 / Music 109A, Spring 2011

Name Solutions Id ______

Each problem is worth 10 points. Round off each decimal approximation to two digits to the right of the decimal.

1. Express each of these musical intervals as an element of $\mathbb{R}^+$ three ways: (1) as a power of 2, (2) as a radical or the reciprical of a radical, and (3) by a decimal approximation.

   (a) up 56 cents $2^{56/1200} = \sqrt[1200]{2^{56}} \cong 1.03$

   (b) down a keyboard fourth $2^{-5/12} = \frac{1}{\sqrt[12]{2^5}} \approx 0.75$

2. Convert to the specified additive measurement the intervals given by the following ratios.

   (a) 7/4, convert to semitones $12 \log_2\left(\frac{7}{4}\right) \approx 9.69$

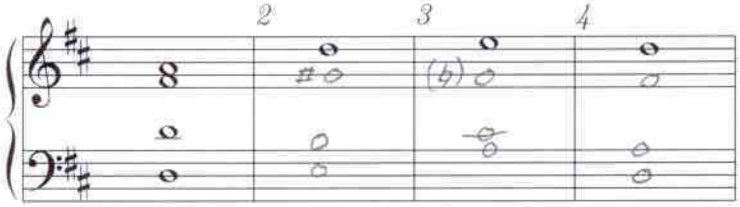
   (b) $\pi^2/5$, convert to cents $1200 \log_2\left(\pi^2/5\right) \approx 1177.28$

3. A string on a stringed instrument has length 100 cm. Indicate the positions of the single fret which will allow the string to play the note (a) a keyboard major third above the original pitch, and (b) a ratio 5/4 above the original pitch.

(a) $2^{4/12} = 2^{1/3} = \frac{F'}{F} = \frac{L}{L'} = \frac{100}{L'}$, so $L' = 100 \cdot 2^{-1/3} \approx 79.37$

(b) $\frac{5}{4} = \frac{F'}{F} = \frac{L}{L'} = \frac{100}{L'}$ so $L' = 100 \cdot \frac{4}{5} = 80$ cm

4. Complete the following to a four-part harmonization of the given melody, major mode, using only whole notes, so that the melody is the top part, the lowest note is always the root, and the result has two parts on each staff. The chords should be the those indicated under the staff.



I II^7 V^7 I

5. Evaluate these logarithms without a calculator. Write down each step of the simplification. You may express your answer as a fraction.

(a) $\log_7\left(\frac{\sqrt[3]{49}}{7}\right) = \log_7 \sqrt[3]{49} - \log_7 7 = \log_7 49^{1/3} - 1 = \log_7 7^{2/3} - 1$
$= \frac{2}{3} - 1 = -\frac{1}{3}$

(b) $\log_b\left(\frac{1}{\sqrt[n]{b^k}}\right) = -\log_b \sqrt[n]{b^k} = -\log_b b^{k/n} = -\frac{k}{n}\log_b b = -\frac{k}{n}$

6. Write on the staff the note which best approximates the frequency having the given interval ratio $r = 3/25$ from the given note. Compute the error in cents.

A_2

$12 \log_2\left(\frac{3}{25}\right) \approx -36.71$
rounds to -37 semitones
with error of 29 cents

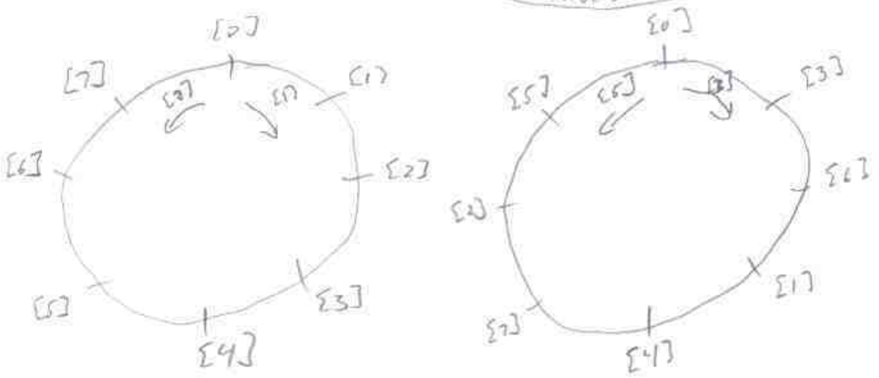
-37 semitones is down
3 octaves and one semitone

7. Determine $\phi(8)$ (ϕ is the Euler phi function) by listing all the generating intervals in the 8-chromatic scale. Indicate which pairs of generating intervals are inverse to each other and for each pair draw the circle of intervals which is based on one element of the pair in the clockwise direction, the other element of the pair in the counterclockwise direction.

Generating intervals are [1], [3], [5], [7], so $\phi(8) = 4$

inverse

inverse



8. Determine whether or not each of the following pairs forms a monoid. If so, is it also also a group? Justify your answers.

(a) $(\mathbb{Q} - \{0\}, \cdot)$ Monoid ✓: identity is 1.
Group ✓: inverse of $\frac{m}{n}$ is $\frac{n}{m}$.

(b) $(\mathbb{Z}, +)$ Monoid ✓: identity is 0
Group ✓: inverse of n is $-n$.

(c) $(\mathbb{Z}_4, \cdot)$ Monoid ✓: identity is [1]
Group ✗: [2], [0] have no inverse.

9. Explain why the functions $f(x) = 2^{x/12}$ and $g(x) = 12\log_2(x)$ are group homomorphisms, and why they are inverse to each other, thereby giving isomorphisms between the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$. Explain the connection of this with musical intervals.

f translates semitones to ratio, g does the reverse.

$f : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, \cdot)$

$f(x+y) = 2^{\frac{x+y}{12}} = 2^{x/12} \cdot 2^{y/12} = f(x) \cdot f(y)$

$g : (\mathbb{R}^+, \cdot) \longrightarrow (\mathbb{R}, +)$

$g(xy) = 12 \log_2 (xy) = 12 [\log_2 x + \log_2 y]$
$= 12 \log_2 x + 12 \log_2 y$
$= g(x) + g(y)$

So f, g are group homomorphisms. Moreover
$f(g(x)) = f(12 \log_2 x) = 2^{\frac{12 \log_2 x}{12}} = 2^{\log_2 x} = x$, and
$g(f(x)) = g(2^{x/12}) = 12 \log_2 2^{x/12} = 12 \cdot \frac{x}{12} \log_2 2$
$= x \log_2 2 = x \cdot 1 = x$.

So f, g are inverse to each other, hence isomorphisms.

10. Create a 6-tone row chart for the original row ([0], [2], [5], [1], [4], [3]) in $\mathbb{Z}_6$

[0]	[2]	[5]	[1]	[4]	[3]
[4]	[0]	[3]	[5]	[2]	[1]
[1]	[3]	[0]	[2]	[5]	[4]
[5]	[1]	[4]	[0]	[3]	[2]
[2]	[4]	[1]	[3]	[0]	[5]
[3]	[5]	[2]	[4]	[1]	[0]