EXAM III

Math 109 / Music 109A, Spring 2009

Name key Id ______

Each problem is worth 10 points.

1. Define a (commutative) ring R and show that $0 \cdot x = 0$, for any $x \in R$.

A ring is a set R with two operations $+$ and $\cdot$ such that $(R, +)$ is an Abelian group and $(R, \cdot)$ is a commutative monoid, and such that $(x+y)z = xz + yz$ for all $x, y, z \in R$. Writing 0 for the identity element for $+$, we have

$$0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x.$$ Subtracting $0 \cdot x$ yields $$0 = 0 \cdot x.$$

2. Prove that the ring $\mathbb{Z}_n$ is an integral domain precisely when n is a prime number.

If n is prime, then if $[m] \cdot [l] = 0$ in $\mathbb{Z}_n$ we have $n | ml$. n is prime so $n | m$ or $n | l$. So $[m] = 0$ or $[l] = 0$. So $\mathbb{Z}_n$ is a domain.

If n is not prime, write $n = lm$ with $0 < l, m < n$. Then $[lm] = [l][m] = 0$, but $[l] \neq 0$ and $[m] \neq 0$, since $n \nmid m$, $n \nmid l$. So $\mathbb{Z}_n$ is not a domain.

3. Prove that there are infinitely many prime numbers.

If $p_1, \cdots, p_n$ were a finite list of all primes, consider $N = p_1 \cdots p_n + 1$. Since $p_i \nmid N$ for all i, N must be prime. But N is not in the list. #

4. Prove that if $y = f(t)$ has period P, then $y = cf(t)$ has period P, but $y = f(ct)$ ($c \neq 0$) has period P/c.

We know $f(t+P) = f(t)$.
Therefore $c f(t+P) = c f(t)$ so $c f(t)$ has period
Also $f(c(t+\frac{P}{c})) = f(ct+P) = f(ct)$, so $f(ct)$ has period $\frac{P}{c}$.

5. On the staff system below, write the keyboard's best approximation for each harmonic up through 11 for the indicated note. For harmonics 5, 7 and 11, indicate how sharp or flat (to the nearest cent) the keyboard's approximation is.

$12 \log_2 5 \approx 27.86$
so keybord is 14 cents sharp

$12 \log_2 7 \approx 33.69$
so keyboard is 31 cents sharp

$12 \log_2 11 \approx 41.51$
so keyboard is 49 cents sharp

6. Find the period, frequency, amplitude, and phase shift for the function

$$g(t) = \sin(880\pi t) + 2\cos(880\pi t)$$

and express it in the form $d\sin(\alpha t+\beta)$, giving a decimal approximation for β.

$P = \frac{1}{440}$ (period)

$2\pi F = 880\pi$, so $F = 440$ (frequency)

$A = 1$, $B = 2$, so $d = \sqrt{1^2+2^2} = \sqrt{5}$ (amplitude)

$$g(t) = \sqrt{5}\left[\frac{1}{\sqrt{5}}\sin(880\pi t) + \frac{2}{\sqrt{5}}\cos(880\pi t)\right]$$

$$= \sqrt{5}\sin(880\pi t + \beta)$$

where $\beta = \arcsin\frac{2}{\sqrt{5}} \approx 1.11$
(phase shift)

7. Suppose a musical tone is sounding A_2, and suppose its second harmonic has amplitude 1/2 and phase shift $\pi/2$. Express the second harmonic in the form $g(t) = A\sin\gamma t + B\cos\gamma t$, for explicit numbers A, B, and γ.

$$F = 110\ Hz \times 2 = 220$$

$$g(t) = \frac{1}{2}\sin\left(2\pi F t + \frac{\pi}{2}\right) = \frac{1}{2}\sin\left(440\pi t + \frac{\pi}{2}\right)$$

$$= \frac{1}{2}\Big[\overbrace{\cos\frac{\pi}{2}}^{=0}\sin(440\pi t) + \overbrace{\sin\frac{\pi}{2}}^{=1}\cos(440\pi t)\Big]$$

$$= \frac{1}{2}\cos(440\pi t) \qquad \left(\text{i.e. } A = 0,\ B = 1/2,\ \gamma = 440\pi\right)$$

8. We established that the square wave, defined on $[0, 2\pi)$ by

$$s(t) = \begin{cases} 1, & \text{for } 0 \le t < \pi \\ -1, & \text{for } \pi \le t < 2\pi \end{cases}$$

has Fourier series

$$s(t) = \sin t + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots$$

Give the values of the Fourier coefficients C, A_k, B_k for $k \in \mathbb{Z}^+$, indicate the phase shift of each harmonic, and explain why the value of B_1 is such.

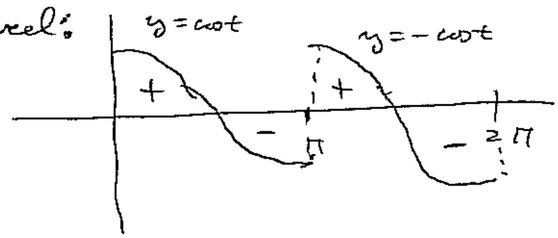
$$C = 0, \quad A_k = \begin{cases} \frac{1}{k} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}, \quad B_k = 0 \text{ for all } k.$$

all phase shifts are 0 (because cosines are absent)

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} (\cos t)\, s(t)\, dt = 0$$

because areas below and above axis cancel:

graph of $(\cos t)s(t)$



9. A certain vowel sound has a formant which amplifies frequencies within 300 Hz of 2500 Hz. A singer sings the vowel at C_3. Which harmonics are amplified? Frequency of C_3 is $\underset{F}{\underset{\shortparallel}{220 \; 2^{-9/12}}} \approx 130.8$

$17 \cdot F \approx 2223.82$
$21 \cdot F \approx 2747.07$
So harmonics 17 thru 21 lie in the range 2200–2800 Hz

10. Express each of these intervals two ways: as rational numbers and in cents, rounding off the latter at 2 digits to the right of the decimal.

(a) the greater whole tone $\frac{9}{8}$ $\quad 1200 \log_2\left(\frac{9}{8}\right) \approx 203.91$ cents

(b) the just minor third $\frac{6}{5}$ $\quad 1200 \log_2\left(\frac{6}{5}\right) \approx 315.64$ cents