EXAM III

Math 109 / Music 109A, Spring 2011

Name solutions Id ____________

Each problem is worth 10 points.

1. Define a (commutative) ring R and show that $0 \cdot x = 0$, for any $x \in R$. Use this to show that $(-1) \cdot x = -x$, for any $x \in R$.

A commutative ring is a set R with two laws of composition $R, +$ and $R, \cdot$, such that $R, +$ is a commutative group and $R, \cdot$ is a commutative monoid, satisfying $a \cdot (b+c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.
For $x \in R$ we have $0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x$. Subtracting $0 \cdot x$ yields $0 = 0 \cdot x$. Also $0 = (1-1) \cdot x = x - 1 \cdot x$ which shows $-1 \cdot x = -x$.

2. Determine whether these subsets of $\mathbb{Z}$ are ideals. If so, express them in the form $n\mathbb{Z}$, where n is a positive integer. Justify your answers.

   (a) the positive integers No. Doesn't contain 0, nor additive inverses

   (b) $100\mathbb{Z} + 30\mathbb{Z}$ Yes.
   $= 10\mathbb{Z}$ since $10 = \gcd(100, 30)$.

3. Prove that there are infinitely many prime numbers. Assume finitely many. Let $p_1, \dots, p_n$ be all the primes. Form $N = p_1 \cdots p_n + 1$. Note that $p_i \nmid N$ since $p_i \nmid 1$. So N has no prime factors, a contradiction.

4. List the units of the ring $\mathbb{Z}_{12}$ and explain the implication this has for keyboard intervals. The units are those $[n]$ for which $\gcd(n, 12) = 1$. Hence $\{[1], [5], [7], [11]\}$. These are precisely the generators for the group $\mathbb{Z}_{12}$, hence the generating keyboard intervals: semitone, fourth, fifth, major seventh.

5. Give the prime factorizations of these integers, writing the primes in ascending order, as in $2^3 \cdot 3^1 \cdot 7^2$.

   (a) 75 $= 3^1 \cdot 5^2$ (b) 64 $= 2^6$ (c) 242 $= 2^1 \cdot 11^2$ (d) 52 $= 2^2 \cdot 13^1$ (e) 14×10^{23} $= 2^{24} \cdot 5^{23} \cdot 7^1$

6. On the staff system below, write the keyboard's best approximation for harmonics 1 through 11 for the indicated note. For the 11th harmonic, indicate how sharp or flat (to the nearest cent) the keyboard's approximation is.

$1200 \log_2 11 \approx 4151.32$

Keyboard approximation is 4200 (3 octaves + tritone), which is ≈ 49 cents sharp.

7. Find the value γ for which the pitch associated to the periodic function $h(t) = d\sin(\gamma t + \beta)$, where t is time in seconds, is $E^\flat_4$.

$\gamma = 2\pi F$ $\qquad F = 440 \cdot 2^{-6/12}$

≈ 1594.87

8. Find the period, frequency, amplitude, and phase shift for the function

$$g(t) = 3\sin(880\pi t) + 4\cos(880\pi t)$$

and express it in the form $d\sin(\alpha t + \beta)$, giving a decimal approximation for β.

$d = \sqrt{3^2+4^2} = \sqrt{25} = 5$ amplitude

$2\pi F = 880\pi$, so $F = 440$ frequency

$P = \frac{1}{440}$ period

$g(t) = 5\left(\frac{3}{5}\sin(880\pi t) + \frac{4}{5}\cos(880\pi t)\right)$

$\frac{4}{5} = \arcsin\beta$, so $\beta \approx 0.93$ phase shift

$g(t) = 5\sin(880\pi t + \beta)$ with β as above.

9. A certain vowel sound has a formant which amplifies frequencies within 350 Hz of 2900 Hz. An alto singer sings the vowel at A_3. Which harmonics are amplified?

$A_3 = 220$

$2550 < 220n < 3250$

$\frac{2550}{220} < n < \frac{3250}{220}$

$\approx 11.6 \qquad \approx 14.8$

so harmonics 12, 13, 14.

10. We established that the square wave, defined on $[0, 2\pi)$ by

$$s(t) = \begin{cases} 1, & \text{for } 0 \le t < \pi \\ -1, & \text{for } \pi \le t < 2\pi \end{cases}$$

has Fourier series

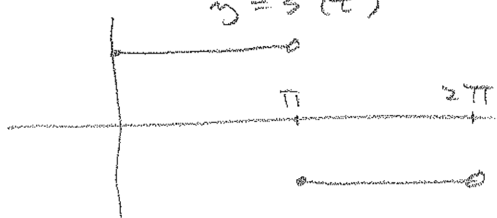
$$s(t) = \sin t + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots$$

Draw the graph of $s(t)$. Give the values of the Fourier coefficients C, A_k, B_k for $k \in \mathbb{Z}^+$, and indicate the amplitude and phase shift of each harmonic. By evaluating an integral using areas, verify that the value of B_1 is what you have determined it to be.

$$C = 0, \quad A_k = \begin{cases} \frac{1}{k} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}, \quad B_k = 0 \text{ for all } k.$$

no cosines, so all phase shifts are 0. amplitude of odd harmonics is $\frac{1}{k}$, of even harmonics is 0.

$y = s(t)$



$$B_1 = \frac{1}{\pi}\int_0^{2\pi} (\cos t)(s(t))\,dt = 0$$

$y = \cos(t)\, s(t)$