FINAL EXAM

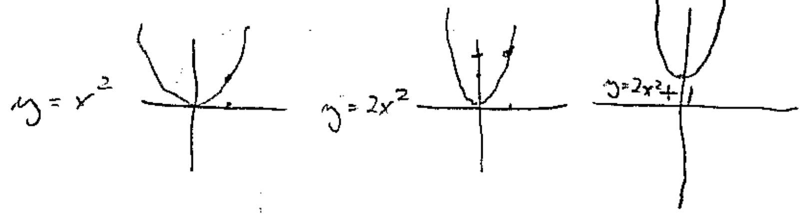
Math 109 / Music 109A, Spring 2009

Name Key Id ____________

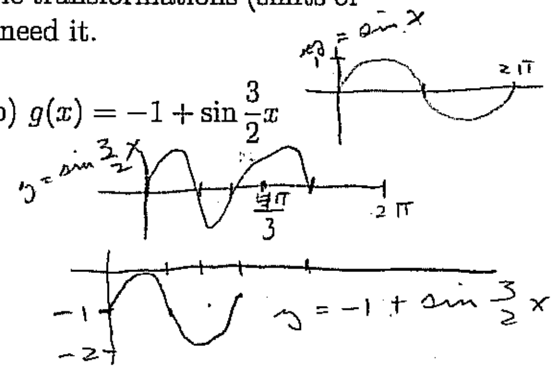
Each problem is worth 10 points.

1. Sketch the graphs of these functions by starting with a more basic function and applying one or more geometric transformations (shifts or stretches). Use the space on page 4 if you need it.

   (a) $f(x) = 2x^2 + 1$



   (b) $g(x) = -1 + \sin \frac{3}{2}x$



2. For the set $\mathbb{Z}$ and a fixed positive integer m, define the equivalence relation whose set of equivalence classes is $\mathbb{Z}_m$. Show that it is in fact an equivalence relation and explain why there are exactly m equivalence classes.

Define $k \equiv l$ to mean $m \mid (k-l)$.

symmetric: If $k \equiv l$, then $k-l = xm$, so $l-k = -xm$ and $m \mid (l-k)$ so $l \equiv k$.

reflexive: $k-k = 0 \cdot m$, so $k \equiv k$.

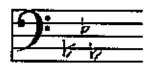
transitive: If $k \equiv l$ and $l \equiv s$ then $k-l = xm$, $l-s = ym$. add these equations to get $k-s = (x+y)m$, showing that $k \equiv s$.

For any $[n] \in \mathbb{Z}_n$, write $n = pm + r$ with $0 \le r < m$. Then $[n] = [r]$ (since $n \equiv r$). This shows $\mathbb{Z}_n = \{[0], \dots, [m-1]\}$. These classes are distinct, for if $0 \le k, l < m$, then $m \nmid k-l$, so $k \not\equiv l$. Hence there are exactly m classes.

3. For the following modes and tonic notes, indicate the appropriate key signature on the given staff, taking note of the clef. Place the sharps and flats in their proper positions.

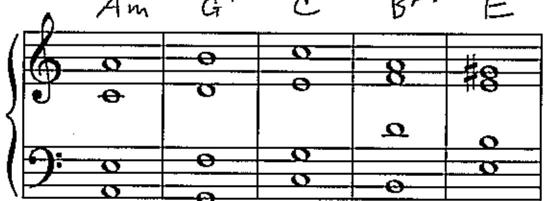
(a) Locrian with tonic D

(b) Myxolydian with tonic B$^\flat$



4. Identify each chord in this minor mode (Aeolian) passage. Above the staff label each chord by root note class with suffix (e.g., E$^{\flat 7}$). Below the staff, label each chord by root scale tone (e.g. $\flat$III^7).

Am   G^7 C   B$^{\text{ø}7}$ E



Im   VII^7 III   II$^{\text{ø}7}$ V

5. (a) Use properties of logarithms to express in terms of logarithms of <u>prime numbers</u>:

$$log_b \left[\left(\frac{20}{\sqrt[3]{21}}\right)^2\right] = 2\left[\log_b 20 - \tfrac{1}{3}\log_b 21\right] = 2\left[2\log_b 2 + \log_b 5 - \tfrac{1}{3}\log_b 7 - \tfrac{1}{3}\log_b 3\right]$$
$$= 4\log_b 2 - \tfrac{2}{3}\log_b 3 + 2\log_b 5 - \tfrac{2}{3}\log_b 7$$

(b) Express as a single logarithm without coefficient, i.e., in the form $\log_b c$:

$$\log_b 15 - \tfrac{1}{2}\log_b 16 = \log_b 15 - \log_b 4 = \log_b \frac{15}{4}$$

6. (a) Express the downward interval of 49 cents as ratio in three ways: as a power of 2, as a radical or the reciprical of a radical, and by a decimal approximation.

$$2^{-49/1200} = \frac{1}{\sqrt[1200]{2^{49}}} \approx 0.97$$

(b) Convert the ratio 7/4 to semitones, rounding off to 2 digits to the right of the decimal and indicating whether the interval is upward or downward. What is the name of this just interval?

$12 \log_2\left(\frac{7}{4}\right) \approx 9.69$ semitones upward

This is the <u>septimal minor seventh</u>

7. Give the (total) duration in beats of:

(a) a sixteenth note in $\frac{12}{8}$ time (compound time signature).

♩. = 1 beat

𝅘𝅥𝅯 = $\boxed{1/6 \text{ beat}}$

(b) a dotted eighth note in $\frac{2}{2}$ time.

𝅗𝅥 = 1 beat

♪ = 1/4 beat

♪. $= \frac{1}{4} \cdot \frac{3}{2} = \boxed{\frac{3}{8} \text{ beat}}$

(c) a quarter note 5-tuplet in $\frac{4}{4}$ time.

$2^2 < 5 < 2^3$ so $r = 2$

$\frac{1}{4} = \frac{1}{2^2}$ So $n + r = 2$ So $\underline{n = 0}$. $\frac{1}{2^0}$-note = whole note, which has duration $\boxed{4 \text{ beats}}$

8. Determine whether or not the pair $(\mathbb{Z}, +)$ is a monoid. If so, is it also also a group? Justify your answer.

$+$ is associative; and 0 is the identity element so <u>$(\mathbb{Z}, +)$ is a monoid</u>.

Moreover, for $n \in \mathbb{Z}$, $-n$ is its inverse, so <u>$(\mathbb{Z}, +)$ is also a group</u>.

9. Complete each excerpt with a measure which repeats the rhythm of the first measure, employing:

(a) diatonic transposition down one scale tone.

(b) chromatic transposition up a minor third.

10. Prove that if $y = f(t)$ has period P, then $y = cf(t)$ has period P, but $y = f(t/c)$ $(c \neq 0)$ has period cP.

$y = f(t)$ has period P says $f(t+P) = f(t)$ for all t.
Let $g(t) = cf(t)$. Then $g(t+P) = cf(t+P) = cf(t) = g(t)$
so g has period P as well.
Let $h(t) = f(\frac{t}{c})$. Then $h(t+cP) = f(\frac{t+cP}{c}) = f(\frac{t}{c}+P) = f(\frac{t}{c})$
$= h(t)$
so h has period cP.

11. Find the period, frequency, amplitude, and phase shift for the function

$$g(t) = 3\sin(440\pi t) + 4\cos(440\pi t)$$

and express it in the form $d\sin(\alpha t+\beta)$, giving a decimal approximation for β.

$d = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$ (amplitude)
$2\pi F = 440\pi$, so $F = 220$ (frequency)
$P = \frac{1}{220}$ (period)
$\beta = \arcsin\frac{4}{5} \approx 0.93$ (phase shift

$g(t) = 5\sin(440\pi t + \beta)$, β as above.

15. Explain the comma of Pythagoras. How does it arise? Evaluate it as a rational number and in cents.

It arises by comparing the complete circle of fifths in just fifths, $\left(\frac{3}{2}\right)^{12}$, with its measurement in octaves, 2^7. The ratio is

$$\frac{\left(\frac{3}{2}\right)^{12}}{2^7} = \frac{3^{12}}{2^{19}}$$

which is $\approx$ 23.46 cents.

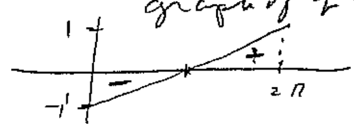
12. The sawtooth wave, defined on $[0, 2\pi)$ by $q(t) = \frac{1}{\pi}t - 1$, has Fourier series

$$q(t) = -\frac{2}{\pi}\left[\sin t + \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) + \frac{1}{4}\sin(4t) + \cdots\right]$$

Give the values of the Fourier coefficients C, A_k, B_k for $k \in \mathbb{Z}^+$, indicate the phase shift of each harmonic, and explain why the value of C is such, based on the graph of $q(t)$.

$$\left.\begin{aligned} C &= 0 \\ A_k &= -\frac{2}{k\pi} \\ B_k &= 0 \end{aligned}\right] \text{ so all phase shifts are } 0.$$

graph of $q(t)$



$$C = \frac{1}{2\pi}\int_0^{2\pi} q(t)\,dt$$

$= 0$ (since area below = area above)

13. The human "ah" vowel has a formant centered at 2640 Hz. A bass singer sings the vowel at G_2. Which harmonics lie within 300 Hz of the center of this formant?

G_2 is $A_2 \cdot 2^{-2/12} = 110 \cdot 2^{-2/12} \approx 98.00$

Harmonics $\boxed{24-30}$ fall in the range 2340 − 2940 Hz.

14. Express each of these just intervals in two ways: as rational numbers and in cents, rounding off the latter at 2 digits to the right of the decimal. What are their keyboard approximations, and how closely are they approximated?

(a) the lesser whole tone $\frac{10}{9}$, ≈ 182 cents, about 18 cents lower than a keyboard step

(b) the just major third $\frac{5}{4}$, ≈ 386 cents, about 14 cents lower than a keyboard major third.