FINAL EXAM

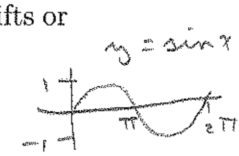
Math 109 / Music 109A, Spring 2011

Name Solutions Id ____________

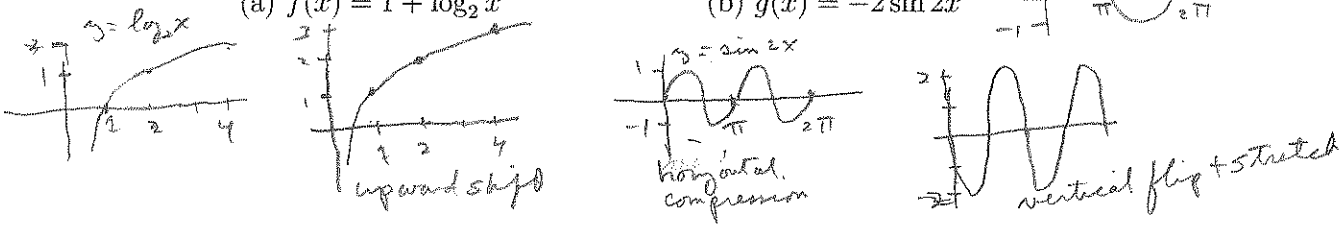
This is an open-book take-home exam. You must work the problems without collaboration. The exam is due by noon on Monday, May 9. You may turn in your completed exam on Monday morning to the office receptionist in the Mathematics Department, Room 100, Cupples I. Alternatively, you may scan the completed exam and email it to Professor Wright at wright@math.wustl.edu.

Each problem is worth 10 points.

1. Sketch the graphs of these functions by starting with a more basic function and applying one or more geometric transformations (shifts or stretches). Use the space on page 4 if you need it.



(a) $f(x) = 1 + \log_2 x$ (b) $g(x) = -2\sin 2x$



2. For the set $\mathbb{Z}$ and a fixed positive integer m, define the equivalence relation whose set of equivalence classes is $\mathbb{Z}_m$. Show that it is in fact an equivalence relation and explain why there are exactly m equivalence classes.

We say $a \equiv b$ if $m \mid (a-b)$, for $a, b \in \mathbb{Z}$

(i) reflexive: $a - a = 0 = 0 \cdot m$, which shows $a \equiv a$ for any $a \in \mathbb{Z}$.

(ii) symmetric: If $a \equiv b$, then $a - b = m \cdot x$ for some $x \in \mathbb{Z}$. Then $b - a = m \cdot (-x)$, which shows $b \equiv a$.

(iii) transitive: assume $a \equiv b$ and $b \equiv c$. Then $a - b = m \cdot x$ and $b - c = m \cdot y$. Adding these equations gives $a - c = a - b + b - c = m \cdot x + m \cdot y = m(x+y)$ which shows $a \equiv c$.

Given any $n \in \mathbb{Z}$, the Div. Alg. gives $n = qm + r$ with $0 \le r < m$. This shows $n \equiv r$. So $[0], [1], \ldots, [m-1]$ are all the classes. If $0 \le k < l \le m-1$, it is impossible for m to divide $k - l$, so $k \not\equiv l$, and $[k] \ne [l]$. So these m classes are distinct.

3. For the following modes and tonic notes, indicate the appropriate key signature on the given staff, taking note of the clef. Place the sharps and flats in their proper positions.

   (a) Locrian with tonic D

   (b) Myxolydian with tonic $B^\flat$

4. Identify each chord in this minor mode (Aeolian) passage. Above the staff label each chord by root note class with suffix (e.g., $E^{\flat 7}$). Below the staff, label each chord by root scale tone (e.g. $\flat III^7$).

Am G^7 C $B^{ø7}$ E

Im VII^7 IIIm $II^{ø7}$ V

5. (a) Use properties of logarithms to express in terms of logarithms of prime numbers:

$$log_b\left[\left(\frac{20}{\sqrt[3]{21}}\right)^2\right] = 2\left[\log_b 20 - \frac{1}{3}\log_b 21\right] = 2\left[\underbrace{\log_b 4}_{2\log_b 2} + \log_b 5\right] - \frac{2}{3}\left[\log_b 3 + \log_b 7\right]$$

$$= 4\log_b 2 - \frac{2}{3}\log_b 3 + 2\log_b 5 - \frac{2}{3}\log_b 7$$

   (b) Express as a single logarithm without coefficient, i.e., in the form $\log_b c$:

$$\log_b 15 - \tfrac{1}{2}\log_b 16 = \log_b 15 - \log_b 4 = \log_b \frac{15}{4}$$

6. (a) Express the downward interval of 49 cents as ratio in three ways: as a power of 2, as a radical or the reciprical of a radical, and by a decimal approximation.

$2^{-49/1200} = \frac{1}{\sqrt[1200]{2^{49}}} \approx 0.97$

(b) Convert the ratio 7/4 to semitones, rounding off to 2 digits to the right of the decimal and indicating whether the interval is upward or downward. What is the name of this just interval?

$12 \log_2 (7/4) \approx 9.69$ semitones; upward, since $\frac{7}{4} > 1$

septimal minor seventh

7. Give the (total) duration in beats of:

(a) a sixteenth note in $\frac{12}{8}$ time (compound time signature). $1/6$ beat

(b) a dotted eighth note in $\frac{2}{2}$ time. $\frac{1}{4}\left(1+\frac{1}{2}\right) = \frac{1}{4} \cdot \frac{3}{2} = \frac{3}{8}$

(c) a quarter note 5-tuplet in $\frac{4}{4}$ time. $2^2 < 5 < 2^3$ so $r = 2$

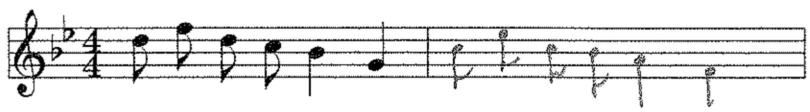
$\frac{1}{2^2} = \frac{1}{2^{n+r}}$ so $n = 0$. $\frac{1}{2^0}$-note is whole note.
So 4 beats.

8. Determine whether or not the pair $(\mathbb{Z}, +)$ is a monoid. If so, is it also also a group? Justify your answer.

$(\mathbb{Z}, +)$ is a monoid, since $+$ is associative and 0 acts as the identity element.
It is a group since $n \in \mathbb{Z}$ has additive inverse $-n$.

9. Complete each excerpt with a measure which repeats the rhythm of the first measure, employing:

   (a) <u>diatonic</u> transposition <u>down</u> one scale tone.



   (b) <u>chromatic</u> transposition <u>up</u> a minor third.

10. Prove that if $y = f(t)$ has period P, then $y = cf(t)$ has period P, but $y = f(t/c)$ $(c \neq 0)$ has period cP.

Let $g(t) = cf(t)$. Then $g(t+P) = c\,f(t+P) = c\,f(t) = g(t)$.

Let $h(t) = f\left(\frac{t}{c}\right)$. Then $h(t+cP) = f\left(\frac{t+cP}{c}\right) = f\left(\frac{t}{c}+P\right)$

$= f\left(\frac{t}{c}\right) = h(t)$.

11. Find the period, frequency, amplitude, and phase shift for the function

$$g(t) = 3\sin(440\pi t) + 4\cos(440\pi t)$$

and express it in the form $d\sin(\alpha t + \beta)$, giving a decimal approximation for β.

$d = \sqrt{3^2+4^2} = 5$ (amplitude)

$2\pi F = 440\pi t$, so $F = 220 = A_3$ (frequency); $\left(\frac{1}{220} = \text{period}\right)$

$g(t) = 5\left[\frac{3}{5}\sin(440\pi t) + \frac{4}{5}\cos(440\pi t)\right]$

$= 5\sin(440\pi t + \beta)$

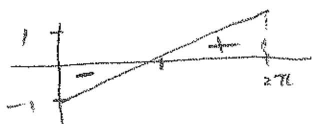
where $\beta = \arcsin\frac{4}{5} \approx 0.93$ (phase shift)

12. The sawtooth wave, defined on $[0, 2\pi)$ by $q(t) = \frac{1}{\pi}t - 1$, has Fourier series

$$q(t) = -\frac{2}{\pi}\left[\sin t + \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) + \frac{1}{4}\sin(4t) + \cdots\right]$$

Give the values of the Fourier coefficients C, A_k, B_k for $k \in \mathbb{Z}^+$, indicate the phase shift of each harmonic, and explain why the value of C is such, based on the graph of $q(t)$.

$C = 0$, $A_k = -\frac{2}{k\pi}$, $B_k = 0$ for all k

graph of $q(t)$ is 

$C = \frac{1}{2\pi}\int_0^{2\pi} q(t)\,dt = 0$ as seen from the graph.

Phase shift: Amplitude β_k is $\sqrt{\left(-\frac{2}{\pi k}\right)^2} = \frac{2}{\pi k}$ and this gives $\sin\beta_k = 0$, $\cos\beta_k = -1$, so $\beta_k = \pi$ for all k.

(Credit was given if you said $\beta_k = 0$, since this situation was not encountered in class.)

13. The human "ah" vowel has a formant centered at 2640 Hz. A bass singer sings the vowel at G_2. Which harmonics lie within 300 Hz of the center of this formant?

G_2 has frequency $F = 110 \cdot 2^{-1/6} \approx 98.00$

$2340 \le nF \le 2940$

$23.9 \le n \le 30$

so harmonics 24 – 30 lie in this range.

14. Express each of these intervals in two ways: as rational numbers expressed by prime factorization $p_1^{\alpha_1}\cdots p_n^{\alpha_n}$, where $\alpha_1\cdots,\alpha_n\in\mathbb{Z}$, and in cents, rounding off the latter at 2 digits to the right of the decimal. What are their keyboard approximations, and how closely are they approximated?

(a) the greater whole tone $\frac{9}{8} = 2^{-3}3^{2}$ ≈ 203.91 cents, $\approx$ 1 step

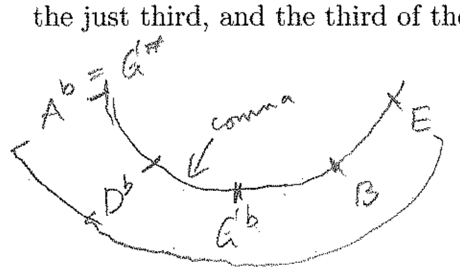
(b) the just major seventh $\frac{15}{8} = 2^{-3}\cdot 3^{1}\cdot 5^{1}$ ≈ 1088.27 cents, $\approx$ major seventh

(c) the comma of Didymus $\frac{81}{80} = 2^{-4}3^{4}5^{-1}$ ≈ 21.51 cents, $\approx$ unison

(d) the mean-tone comma $\frac{128}{125} = 2^{7}5^{-3}$ ≈ 41.06 cents, $\approx$ unison

15. (a) Explain the comma of Pythagoras. How does it arise? Express it as a rational number and in cents. It is the overshoot of going around the circle of fifths with 12 just (3/2) fifths. This overshoots 7 octaves by $\left(\frac{3}{2}\right)^{12}\frac{1}{2^7} = \frac{3^{12}}{2^{19}}$. In cents this is ≈ 23.46

(b) Suppose a keyboard has Pythagorean tuning, with the comma extracted between $G^\flat$ and $D^\flat$. How does this keyboard render the major third from E to $G^\sharp$? Compare this third to the usual Pythagorean third, the just third, and the third of the usual equally tempered keyboard.



This third is 4 just fifths − comma of Pythagoras − 2 octaves

i.e. $\left(\frac{3}{2}\right)^{4}\frac{2^{19}}{3^{12}}\frac{1}{2^{2}} = \frac{2^{13}}{3^{8}} \approx 384.36$ cents

just 3rd $\frac{5}{4}$ is ≈ 386.31 cents

Pythagorean 3rd $\frac{3^4}{2^6}$ is ≈ 407.82 cents

tempered 3rd is 400 cents

This third is about 2 cents under just 3rd, 16 cents under tempered third, 23 cents under Pythagorean 3rd.