Homework 5

Math 109 / Music 109A, Spring 2009

Due Monday, March 30.

1. Prove that in any (commutative) ring R we have $(-1) \cdot x = -x$ and $0 \cdot x = 0$, for any $x \in R$.

2. Express each of these ideals in $\mathbb{Z}$ in the form $n\mathbb{Z}$, where n is a positive integer:

   (a) $12\mathbb{Z} + 15\mathbb{Z}$ (b) $5\mathbb{Z} + (-20)\mathbb{Z}$

   (c) $10\mathbb{Z} + 44\mathbb{Z}$ (d) $13\mathbb{Z} + 35\mathbb{Z}$

3. Prove that there are infinitely many prime numbers. (Hint: If $p_1, \ldots, p_n$ were a complete list of primes, consider a prime factor of $p_1 \cdots p_n + 1$.)

4. Prove that if p is prime and $n \in \mathbb{Z}$, then either $p \mid n$ or $\gcd(p, n) = 1$.

5. Give the prime factorizations of these integers, writing the primes in ascending order, as in $2^3 \cdot 3 \cdot 7^2$.

   (a) 110 (b) 792 (c) 343 (d) 3422 (e) 15×10^{23}

6. List the primes larger than 13 but less than 50, and for each, determine how closely its musical interval is approximated by the keyboard, calculating the error in cents. Which primes have good keyboard approximations, where "good" means within 15 cents?

7. Call a musical interval a *prime interval* if its interval ratio is a prime integer; call it a *rational interval* if its interval ratio is a rational number.

Show that all rational intervals can be written as compositions of prime intervals and their opposites.

8. Given $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, prove that $[n]$ is a generator for $\mathbb{Z}_m$ if and only if $\gcd(m, n) = 1$. Interpret this as a statement about generating intervals in the modular m-chromatic scale.

9. Prove that the ring $\mathbb{Z}_n$ is an integral domain precisely when n is a prime number.

10. Compose a brief melodic passage using one of the m on n techniques discussed at the end of this Chapter 8. Just a very few bars will do.