1. First notice that
\[ \Delta = \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}. \]
Since
\[ \Delta(zh) = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial(zh)}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \left( h + \frac{\partial h}{\partial z} \right) = 4 \frac{\partial h}{\partial \bar{z}} + \Delta h, \]
it follows that \( \partial h/\partial \bar{z} = 0 \), i.e. \( h \) is holomorphic. \( \square \)

2. Every point of \( \mathbb{C} \setminus \{0\} \) has a neighborhood where \( \log z \) has a single-valued branch. Therefore, \( \log |z| \) is harmonic in such neighborhood, being the real part of a holomorphic function. Furthermore, \( \log z \) is holomorphic in \( \mathbb{C} \setminus ((-\infty, 0]) \), which means that its imaginary part \( \text{Im} \log z = \arg z \) is a conjugate harmonic function to \( \log |z| \).

Suppose that \( \log |z| \) has a conjugate harmonic function on \( \mathbb{C} \setminus \{0\} \), say \( v(z) \). Since the conjugate harmonic function is unique up to a constant, there is \( C \in \mathbb{R} \) such that \( v(z) = \arg z + C \) for all \( z \in \mathbb{C} \setminus ((-\infty, 0]) \). But then
\[ \lim_{y \downarrow 0} v(-1 \pm iy) = C + \lim_{y \downarrow 0} \arg(-1 \pm iy) = C \pm \pi, \]
which means that \( v \) is discontinuous at \(-1\), a contradiction. \( \square \)

3. Let \( f_1(z) = iBA^{-1}z \) and \( f_2(z) = e^z \). Obviously \( f_1 \) maps the strip \( D = \{ z : |\text{Re} \, z| < A \} \) bijectively onto a horizontal strip \( D_1 = \{ z : |\text{Im} \, z| < B \} \). Using the formula \( e^{x+iy} = e^x e^{iy} \), we see that \( f_2 \) maps \( D_1 \) onto the set \( D_2 = \{ z : |\arg z| < B \} \). The mapping \( f_2 \) is also bijective, see Problem 2 of Homework 2. Thus \( z \mapsto \exp(iBA^{-1}z) \) is a bijective holomorphic (hence conformal) mapping from \( D \) onto \( D_2 \). \( \square \)

4. A line that passes through the origin can be parametrized as follows: \( L = \{ te^{i\theta} : t \in \mathbb{R} \} \). Then the image of \( L \) under the map \( w = 1/z \) is \( \{ t^{-1} e^{-i\theta} : t \in \mathbb{R} \} \), which is another line passing through the origin.

Now suppose that \( L \) does not pass through 0. Let \( \zeta \) be the point symmetric to 0 with respect to \( L \). Then \( L = \{ z : |z - \zeta| = |z| \} \) is the set of points that are equidistant from 0 and \( \zeta \). Therefore, the image of \( L \) under inversion is
\[ \{ w : |1/w - \zeta| = |1/w| \} = \{ w : |w - 1/\zeta| = 1/|\zeta| \}, \]
which is a circle centered at \( 1/\zeta \). \( \square \)

5. (a) Suppose that \( f(z) = (az + b)/(cz + d) \) is not the identity map. Then its finite fixed points are precisely the roots of the non-zero polynomial \( q(z) = az + b - z(cz + d) \). If \( c \neq 0 \), then \( q \) has 1 or 2 roots, and those are the only fixed points of \( f \), because \( f(\infty) = a/c \neq \infty \). Suppose that \( c = 0 \). Then \( q \) has one root if \( a - d \neq 0 \), and no roots otherwise. Since \( f(\infty) = \infty \), the map \( f \) has either 1 or 2 fixed points.
(b) Let \( z_1, z_2 \) be the (only) fixed points of \( f \). It is easy to construct a fractional linear transformation \( g \) such that \( g(0) = z_1 \) and \( g(\infty) = z_2 \). Now the composition \( h = g^{-1} \circ f \circ g \) fixes 0 and \( \infty \), which implies \( h(z) = az \) for all \( z \). Here \( a \neq 0, 1 \) because \( f \) is invertible and is not the identity map.

Suppose that two maps \( f_1(z) = a_1 z \) and \( f_2(z) = a_2 z \) are conjugate by a fractional linear transformation \( g \), that is, \( g^{-1} \circ f_1 \circ g = f_2 \). Then \( f_1 \circ g = g \circ f_2 \), which means that

\[
(1) \quad a_1 g(z) = g(a_2 z), \quad z \in \mathbb{C}.
\]

Suppose that \( g(0) \neq \infty \). Since both sides of (1) must have the same derivative at \( z = 0 \), it follows that \( a_1 g'(0) = a_2 g'(0) \). But \( g'(0) \neq 0 \) because \( g \) is a conformal map. Thus \( a_1 = a_2 \). If \( g(0) = \infty \), then the map \( g_1(z) = 1/g(z) \) is holomorphic in a neighborhood of 0, and \( a_1^{-1}g_1(z) = g_1(a_2 z) \). Equating derivatives at \( z = 0 \) as above, we obtain \( a_1^{-1} = a_2 \).

Conversely, if \( a_1^{-1} = a_2 \), then \( f_1 \) and \( f_2 \) are conjugate by \( z \mapsto 1/z \).

In conclusion, two distinct maps \( z \mapsto a_1 z \) and \( z \mapsto a_2 z \) are conjugate if and only if \( a_1 a_2 = 1 \).

(c) Let \( z_1 \) be the only fixed point of \( f \). There is a fractional linear transformation \( g \) such that \( g(\infty) = z_1 \). The composition \( h = g^{-1} \circ f \circ g \) fixes \( \infty \) and no other point. This is only possible if \( h(z) = z + b \), \( b \neq 0 \) (see (a) above). Let \( p(z) = bz \); then \( p^{-1} \circ h \circ p(z) = b^{-1}(bz + b) = z + 1 \), as required.

In a summary, every fractional linear transformations is conjugate to exactly one of the following maps:

(i) the identity;
(ii) \( z \mapsto az \), where either \( |a| > 1 \) or \( a = e^{i\theta} \) with \( 0 < \theta \leq \pi \).
(iii) \( z \mapsto z + 1 \).