Due Wednesday, October 29.

1. Suppose $f(z) = \sum a_k z^k$ and $g(z) = \sum b_k z^k$ are holomorphic at $z = 0$ having radii of convergence $R$ and $S$ respectively.

   (a) Prove that $f(z)g(z)$ is represented by the power series $\sum c_k z^k$ where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$, for $k \geq 0$, converging for $|z| < \min(R, S)$.

   (b) Give an example of an $f(z)$ and $g(z)$ where $R \neq S$ and the radius of convergence of $\sum c_k z^k$ is strictly greater than both $R$ and $S$.

2. Define the Bernoulli numbers $B_n$ by

   $$\frac{z}{2} \cot(z/2) = 1 - B_1 \frac{z^2}{2!} - B_2 \frac{z^4}{4!} - B_3 \frac{z^6}{6!} - \cdots.$$  

   Find the radius of convergence and compute the first five Bernoulli numbers.

3. Show that the coefficients of a power series “depend continuously” on the function they represent, in the following sense: If $\{f_m(z)\}$ is a sequence of holomorphic functions converging normally to $f(z)$ for $|z| < \rho$, and

   $$f_m(z) = \sum a_{k,m} z^k, \quad f(z) = \sum a_k z^k,$$

   then for each $k \geq 0$, we have $a_{k,m} \to a_k$ as $m \to \infty$.

4. Show that if a holomorphic function $f(z)$ has a zero of order $N$ at $z_0$, then $f(z) = g(z)^N$ for some $g(z)$ holomorphic near $z_0$ and satisfying $g'(z_0) \neq 0$.

5. Show that if $f(z)$ is a nonconstant holomorphic function on a domain $D$, then the image under $f$ of any open set is open. **Remark:** This is the Open Mapping Theorem for holomorphic functions. It suffices to show any $z \in D$ has an arbitrarily small open neighborhood whose image is open. If $f'(z) \neq 0$ use the Local Inverse Function Theorem. If $f'(z) = 0$ use Exercise \#4.