1. The complex Fourier series \( \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \) has coefficients \( c_k = \frac{i}{2\pi} \int_{-\pi}^{\pi} \theta e^{-ik\theta} d\theta \). It immediately follows that \( c_0 = 0 \). For \( k \neq 0 \) integration by parts yields
\[
c_k = \frac{i}{2\pi} \theta e^{-ik\theta} \bigg|_{-\pi}^{\pi} + \frac{1}{2\pi i k} \int_{-\pi}^{\pi} e^{-ik\theta} d\theta = \frac{ie^{i\theta}}{k} = \frac{(-1)^k i}{k}.
\]
Since the given function is odd, we can obtain its sine series by formally rearranging the complex Fourier series.
\[
\sum_{k \neq 0} \frac{(-1)^k i}{k} e^{ik\theta} = \sum_{k=1}^{\infty} \frac{(-1)^k i}{k} (e^{ik\theta} - e^{-ik\theta}) = \sum_{k=1}^{\infty} \frac{(-1)^k i}{k} 2i \sin k\theta = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin k\theta.
\]
When \( \theta = \pm \pi \), the complex Fourier series diverges because
\[
\sum_{k \neq 0} \frac{(-1)^k i}{k} e^{\pm i\pi} = \sum_{k \neq 0} \frac{(-1)^k i}{k} (-1)^k = \sum_{k \neq 0} \frac{i}{k},
\]
and the harmonic series diverges. On the other hand, every term of the sine series vanishes when \( \theta = \pm \pi \), hence the series converges.

Finally, the formal derivative of the complex Fourier series is
\[
\sum_{k \neq 0} \frac{(-1)^k i}{k} ike^{ik\theta} = \sum_{k \neq 0} (-1)^{k+1} e^{ik\theta}.
\]
This series diverges for every \( \theta \), since its terms do not approach 0. \( \square \)

2. (a) \( e^{1/z} \) has an essential singularity at 0. From its Laurent expansion
\[
e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}, \quad z \neq 0,
\]
we obtain \( \text{res}_{z=0} e^{1/z} = 1 \).

(b) \( \tan z = \sin z / \cos z \) has simple poles at \( z_k = \pi/2 + \pi k, \; k \in \mathbb{Z} \) (the poles are simple because \( (\cos z)' = \pm 1 \neq 0 \) at those points.) Thus
\[
\text{res}_{z=z_k} \tan z = \frac{\sin z_k}{(\cos z)'|_{z=z_k}} = \frac{\sin z_k}{-\sin z_k} = -1.
\]

(c) The function \( f(z) = z(z^2 + 1)^{-2} \) has poles or order 2 at \( \pm i \). Therefore,
\[
\text{res}_{z=\pm i} f = \left. \frac{d}{dz} ((z \mp i)^2 f(z)) \right|_{z=\pm i} = \left. \frac{d}{dz} (z(z \pm i)^{-2}) \right|_{z=\pm i} = \left. (z \pm i)^{-2} - 2z(z \pm i)^{-3} \right|_{z=\pm i} = (-2i)^{-2} \mp 2i(-2i)^{-3} = -1/4 + 1/4 = 0.
\]

(d) Since
\[
f(z) = \frac{1}{z^2 + z} = \frac{1}{z(z + 1)} = \frac{1}{z} - \frac{1}{z + 1},
\]
it follows that \( \text{res}_{z=0} f = 1 \) and \( \text{res}_{z=-1} f = -1 \).
3. (a) Let $D_R = \{z : |z| < R, \text{Im} \, z > 0\}$ be the half-disk bounded by the segment $[-R, R]$ and half-circle $\gamma_R$, $R > 1$. The function $f(z) = z^2(z^4 + 1)^{-1}$ has two simple poles in $D_R$, namely $\zeta$ and $i\zeta$, where $\zeta = e^{\pi i/4} = (1 + i)/\sqrt{2}$. We have

$$\text{res}_{z=\zeta} f = \frac{\zeta^2}{4\zeta^3} = \frac{1}{4\zeta}, \quad \text{res}_{z=i\zeta} f = \frac{(i\zeta)^2}{4(i\zeta)^3} = -\frac{i}{4\zeta}.$$ 

Therefore,

$$\int_{\partial D_R} f(z)dz = 2\pi i(\text{res}_{z=\zeta} f + \text{res}_{z=i\zeta} f) = \frac{2\pi i(1 - i)}{4\zeta} = \frac{\pi i\sqrt{2}e^{-\pi i/4}}{2e^{\pi i/4}} = \frac{\pi i}{\sqrt{2}e^{\pi i/2}} = \frac{\pi}{\sqrt{2}}.$$ 

Since

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \pi R \max_{\gamma_R} |f(z)| = \pi R \frac{R^2}{R^4 - 1} \to 0, \quad R \to \infty,$$

it follows that $\lim_{R \to \infty} \int_{-R}^R f(x)dx = \pi/\sqrt{2}$, hence $\int_{-R}^R f(x)dx = \pi/(2\sqrt{2})$.

(b) Let $D_R$ and $\gamma_R$ be as above. Since $\sin z = (e^{iz} - e^{-iz})/(2i)$ grows exponentially in the upper half-plane, $\int_{\gamma_R} \sin^2 z(z^2 + 1)^{-1}dz$ might not approach 0 as $R \to \infty$. We can overcome this difficulty by constructing an appropriately decaying holomorphic function whose real part is equal to $\sin^2 z(z^2 + 1)^{-1}$ for $z \in \mathbb{R}$. Indeed, for $z \in \mathbb{R}$ we have

$$\sin^2 z = \frac{1 - \cos 2z}{2} = \frac{\text{Re}(1 - e^{2iz})}{2}.$$ 

Unlike $\sin^2 z$, the function $(1 - e^{2iz})/2$ is bounded (by 1) in the upper half-plane. Indeed, $\text{Im} \, z > 0$ implies the estimate $|e^{2iz}| = e^{\text{Re}2iz} = e^{-2\text{Im} \, z} < 1$. So, let

$$f(z) = \frac{1 - e^{2iz}}{2(z^2 + 1)}.$$ 

This function has one simple pole $i$ in $D_R$. Since

$$\text{res}_{z=i} f = \frac{1 - e^{2i}}{4i} = \frac{1 - e^{-2}}{4i},$$

it follows that $\int_{\partial D_R} f(z)dz = 2\pi i \text{res}_{z=i} f = \pi(1 - e^{-2})/2$. Since

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \pi R \max_{\gamma_R} |f(z)| \leq \pi R \frac{2}{2(R^2 - 1)} \to 0, \quad R \to \infty,$$

we conclude that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2 + 1}dx = \lim_{R \to \infty} \int_{-R}^R \text{Re} \, f(x)dx = \text{Re} \lim_{R \to \infty} \int_{-R}^R f(x)dx = \frac{\pi(1 - e^{-2})}{2}.$$ 

4. (a) Given $R > 1$ and $0 < \varepsilon < 1/R$, consider domains $A = \{z : 1/R < |z| < R\}$ and $B = \{z : \text{Re} \, z > 0, |\text{Im} \, z| < \varepsilon\}$. Their difference $D = A \setminus \overline{B}$ is bounded by two circular arcs $\gamma = \{z : |z| = 1/R\} \setminus B$, $\Gamma = \{z : |z| = R\} \setminus B$ and two segments $I_1 = A \cap \{z : \text{Re} \, z > 0, \text{Im} \, z = \varepsilon\}$, $I_2 = A \cap \{z : \text{Re} \, z > 0, \text{Im} \, z = -\varepsilon\}$. The function $z^{-a}$
has a holomorphic branch in $D$, namely, $(re^{i\theta})^{-a} = r^{-a}e^{-ia\theta}$, $0 < \theta < 2\pi$. Consequently, one can apply the residue theorem to $f(z) = z^{-a}/(1 + z)$ to obtain
\[
\int_{\partial D} f(z)dz = 2\pi i \text{res}_{z=-1} f = 2\pi i z^{-a}_{|z|=1} = 2\pi ie^{-\pi ia}.
\]
Since $f$ is uniformly continuous in $\overline{D}$, it follows that
\[
\int_{I_1} f(z)dz \to \int_{1/R}^{R} \frac{x^{-a}}{1 + x}dx, \quad \int_{I_2} f(z)dz \to -e^{-2\pi i a} \int_{1/R}^{R} \frac{x^{-a}}{1 + x}dx,
\]
as $\varepsilon \to 0$. Furthermore,
\[
\left|\int_{\gamma} f(z)dz\right| \leq 2\pi R^{-1} \max_{\gamma} |f(z)| = 2\pi R^{a-1}/(1 - 1/R) \to 0, \quad R \to \infty;
\]
\[
\left|\int_{\Gamma} f(z)dz\right| \leq 2\pi R \max_{\Gamma} |f(z)| = 2\pi R^{1-a}/(R - 1) \to 0, \quad R \to \infty.
\]
Combining the above yields
\[
\int_{0}^{\infty} \frac{x^{-a}}{1 + x}dx = \frac{2\pi i e^{-\pi ia}}{1 - e^{-2\pi ia}} = \frac{\pi}{\sin(\pi a)}.
\]
(b) For $R > 1$ let $D_R = \{re^{i\theta} : 0 < r < R, 0 < \theta < 2\pi/b\}$. The function $z^b$ has a holomorphic branch in $D_R$, namely, $(re^{i\theta})^b = r^b e^{ib\theta}$, $0 < \theta < 2\pi/b$. Since $f(z) = 1/(1 + z^b)$ continuously extends to $\overline{D}$, it follows that
\[
\int_{\partial D} f(z)dz = 2\pi i \text{res}_{z=\exp\{\pi i/b\}} f = \frac{2\pi i}{b \exp\{\pi i(b - 1)/b\}} = -2\pi b^{-1}ie^{\pi i/b}.
\]
The boundary of $D$ consists of $I_1 = [0, R], I_2 = \{re^{2\pi i/b} : 0 \leq r \leq R\}$, and a circular arc $\Gamma_R = \{Re^{i\theta} : 0 \leq \theta \leq 2\pi/b\}$. It is easy to see that $i_{I_2} f(z)dz = -e^{2\pi i/b} \int_{I_1} f(z)dz$. Since
\[
\left|\int_{\Gamma_R} f(z)dz\right| \leq 2\pi b^{-1}R \max_{\Gamma_R} |f(z)| = 2\pi b^{-1}R/(R^b - 1) \to 0, \quad R \to \infty,
\]
we obtain
\[
\int_{0}^{\infty} \frac{dx}{1 + x^b} = \frac{-2\pi ib^{-1}e^{\pi i/b}}{1 - e^{2\pi i/b}} = \frac{\pi}{b \sin(\pi/b)}.
\]
Finally, the change of variables $x^b = u$ yields
\[
\frac{\pi}{b \sin(\pi/b)} = \int_{0}^{\infty} \frac{dx}{1 + x^b} = \int_{0}^{\infty} \frac{u^{1/b-1}}{b(1 + u)} du = \frac{\pi}{b \sin(\pi a)},
\]
where $a = 1 - 1/b$. Since $\sin(\pi(1 - 1/b)) = \sin(\pi/b)$, the above results indeed match.

5. For $R > 1$ let $D_R = \{z : |z| < |z| < R, \text{Im} z > 0\}$. The function $f(z) = (e^{2iz} - 1)/z^2$ is holomorphic in $D_R$, hence $\int_{\partial D} f(z)dz = 0$. The boundary of $D$ consists of two arcs ($\gamma_R = \{z : |z| = 1/R, \text{Im} z \geq 0\}, \Gamma_R = \{z : |z| = R, \text{Im} z \geq 0\}$) and two segments
\([-R, -1/R], [1/R, R]\)). By the fractional residue theorem the integral of \(f\) over \(\gamma_R\) (oriented clockwise) is equal to \(-\pi i \text{res}_{z=0} f = -\pi i(2i) = 2\pi\), where the residue is found from the Laurent expansion \(f(z) = 2i/z + (2i)^2/2 + \ldots\). On the other hand,

\[
\left| \int_{\Gamma} f(z) \, dz \right| \leq \pi R \max_{\Gamma_R} |f(z)| \leq 2\pi R/R^2 \to 0, \quad R \to \infty,
\]

(recall that \(|e^{2iz}| < 1\) in the upper half-plane). It follows that

\[
\int_{-1/R}^{-R} f(x) \, dx + \int_{1/R}^{R} f(x) \, dx \to -2\pi, \quad R \to \infty,
\]

even though \(\int_{-\infty}^{\infty} f(x) \, dx\) diverges. Furthermore, for any \(x \in \mathbb{R} \setminus \{0\}\)

\[
\text{Re } f(x) = \frac{\cos(2x) - 1}{x^2} = -2 \frac{\sin^2 x}{x^2}.
\]

Therefore,

\[
\int_{-1/R}^{-R} \frac{\sin^2 x}{x^2} \, dx + \int_{1/R}^{R} \frac{\sin^2 x}{x^2} \, dx \to \pi, \quad R \to \infty.
\]

Since \(\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx\) is a convergent integral, we conclude that

\[
\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx = \pi.
\]