Mid-Term Exam, Math 421, Fall 2003

Work all problems.

1. Find a 1–1 holomorphic function from the 1st quadrant $\text{Re } z > 0$, $\text{Im } z > 0$ onto the vertical strip $-1 < \text{Re } z < 1$.

2. State the general version of Cauchy Integral Theorem, the one giving a formula for $f^{(m)}(z)$. Prove the formula for $m = 0$, using Cauchy’s Theorem.

3. Prove that the points $z_1, z_2, z_3$ in the complex plane are vertices of an equilateral triangle if and only if $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_1 z_3 - z_2 z_3 = 0$.

4. Find a fractional linear transformation which takes the upper half plane to the interior of the unit circle centered at 0, and which fixes the points $z = \pm 1$.

5. Show that a holomorphic function is harmonic, but that a complex-valued harmonic function is not necessarily holomorphic.

6. Evaluate:
   
   (a) $\int_{|z|=\frac{1}{2}} \frac{dz}{z^3 - 1}$
   
   (b) $\int_{|z|=2} \frac{1}{1 + z + z^2 + z^3} dz$
   
   (c) $\int_{|z|=4} \frac{\sin \frac{z}{2}}{z^3 - \pi z^2} dz$