Homework 2, Math 422, Spring 2004

Due Wednesday, February 11.

1. Find the Cauchy integrals of the following functions around the unit circle \(|z| = 1\), positively oriented: (a) \(z\)  
(b) \(1/z\)  
(c) \(\text{Re}(z)\)  
(d) \(\text{Im}(z)\)

2. Show that a domain \(D\) in the complex plane is simply connected if and only if any holomorphic function \(f(z)\) on \(D\) which is nonvanishing on \(D\) has a holomorphic logarithm on \(D\). Hint: Consider \(G(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)} \, dw\).

3. Let \(f(z)\) be holomorphic for \(|z - z_0| < R\), with \(f(z_0) = w_0\) and \(|f(z) - w_0| \leq M\) for \(|z - z_0| < R\). Show that if \(f(z)\) has a critical point of order \(m - 1\) at \(z_0\), then
\[
|f(z) - w_0| \leq \frac{M}{R^m} |z - z_0|^m
\]
for \(|z - z_0| < R\), with equality holding at some point \(z' \neq z_0\) if and only if
\[
f(z) = w_0 + \kappa(z - z_0)^m
\]
for some \(\kappa \in \mathbb{C}\) with \(|\kappa| = \frac{M}{R^m}\).

4. Let \(D = \mathbb{C} - \{a_1, \ldots, a_m\}\) be the complex plane with \(m\) punctures. Show that any conformal self-map of \(D\) is a fractional linear transformation that permutes \(\{a_1, \ldots, a_m, \infty\}\).

5. Suppose \(f(z)\) is a holomorphic function from \(\mathbb{D}\) to itself that is not the identity map. Show that \(f(z)\) has at most one fixed point.