1. Since \( f(z) \) and \( g(z) \) are holomorphic on \( D \cup \partial D \), we apply the Green’s theorem. Notice \( \frac{\partial f(z)}{\partial z} = f(z) \frac{\partial g(z)}{\partial \bar{z}} + g(z) \frac{\partial f(z)}{\partial \bar{z}} \). From the Green Theorem

\[
\int_{\partial D} f(z)g(z)dz = 2i \int_{D} \frac{\partial f(z)}{\partial \bar{z}} g(z)dx dy = 2i \int_{D} f(z)g'(z)dx dy
\]

2. (a) For the hyperbolic disk, we know \( u(\rho) \) maps, so \( u(\rho) \) and has the real part \( \Re u \). Since \( u(z) \) is defined on \( \bar{D} \), the inverse of \( f \) is also a FLT, so \( f^{-1}(f(D(z_0, \rho))) = D(z_0, \rho) \) (the interior get mapped to each other) is a Euclidean disk.

3. The Poisson kernel is defined as \( P_r(\theta) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta} \) and it converges normally on \( r \in [0,1] \). Then the Fourier coefficient is given by \( c_l = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta} e^{-il\theta} \frac{dl}{2\pi} = \sum_{k=-\infty}^{\infty} r^{|k|} \frac{f^\pi}{l} e^{ilk\theta} e^{-il\theta} \frac{dl}{2\pi} = \sum_{k=-\infty}^{\infty} r^{|k|} \delta_{kl} = r^{|l|} \)

4. Since \( u \) is defined on \( \bar{D} \), we can use these boundary values to extend it to \( u'(z) = \int_{0}^{2\pi} u(e^{i\phi}) Re(\frac{e^{i\phi} + z}{e^{i\phi} - z}) \frac{d\phi}{2\pi} \).

Since \( u - u' \) is a harmonic function that is zero on \( \partial D \), the Maximum Modulus Principle states that \( u - u' \equiv 0 \) on \( D \), so \( u = u' \). Consider function \( g(z) = \int_{0}^{2\pi} u(e^{i\phi}) \frac{e^{i\phi} + z}{e^{i\phi} - z} \frac{d\phi}{2\pi} \), it is holomorphic on \( D \) and has the real part \( u \), so \( f(z) \) differs from \( g(z) \) by one constant. Say \( f(z) = \int_{0}^{2\pi} u(e^{i\phi}) \frac{e^{i\phi} + z}{e^{i\phi} - z} \frac{d\phi}{2\pi} + c \), \( f(0) = \int_{0}^{2\pi} u(e^{i\phi}) \frac{d\phi}{2\pi} + c \equiv u(0) + c \) together with the definiton \( f(z) = u(z) + iv(z) \), we have that \( c = iv(0) \). So the result follows.

5. a function \( u \) is harmonic if it satisfies the mean value property. So, we need to show \( G(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} G(z_0 + re^{i\theta})d\theta \) for all \( z_0 \in D \) such that \( D(z_0, r) \subset D \). Given above \( z_0 \) and \( r \), \( \frac{1}{2\pi} \int_{0}^{2\pi} G(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} g(t, z_0 + re^{i\theta})dt d\theta = \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} g(t, z_0 + re^{i\theta})d\theta dt = \int_{0}^{\pi} g(t, z_0)dt = G(z_0) \)

Notice the above double integral is interchange because \( g(t, z_0 + re^{i\theta}) \in L^1(dt \times d\theta)(t,\theta \text{ defined on the compact set together with } g \text{ continuous imply } g(t, z_0 + re^{i\theta}) \) is bounded at \([a, b] \times [0, 2\pi]\), so \( g(t, z_0 + re^{i\theta}) \) is integrable), then we apply Fubini-Tonelli theorem.

March 1, 2004

HW3 solution, Math 422