Homework 4, Math 422, Spring 2004

Due Wednesday, March 3.

1. A finite Blaschke Product is a rational function of the form

\[ B(z) = e^{i\varphi} \left( \frac{z - a_1}{1 - \overline{a_1}z} \right) \cdots \left( \frac{z - a_n}{1 - \overline{a_n}z} \right), \]

where \( a_1, \ldots, a_n \in \mathbb{D} \) and \( 0 \leq \varphi < 2\pi \). Show that if \( f(z) \) is continuous for \( |z| \leq 1 \) and holomorphic for \( |z| < 1 \), and if \( |f(z)| = 1 \) for \( |z| = 1 \), then \( f(z) \) is a finite Blaschke product.

2. Show that if \( f(z) \) is meromorphic for \( |z| < 1 \), and \( |f(z)| \to 1 \) as \( |z| \to 1 \), then \( f(z) \) is a rational function, and is, more specifically, the quotient of two finite Blaschke products.

3. Suppose the curve \( \gamma \) passing through 0 is the graph of a function \( y = h(x) \) that can be expressed as a convergent power series \( h(x) = \sum_{k=1}^{\infty} a_k x^k \), \(-r < x < r\), where the \( a_k \)'s are real. (a) Show that \( z = \zeta + ih(\zeta) \) can be solved for \( \zeta \) as a holomorphic function of \( z \) for \( |z| < \epsilon \). (b) Show that \( \gamma \) is an analytic arc near 0. (c) Show that the reflection through \( \gamma \) is given by \( z^* = \overline{h(\overline{z})} - \overline{\zeta} \).

4. Determine the reflection \( z \mapsto z^* \) across the parabola \( y = x^2 \). Expand \( z^* \) in a power series, in powers of \( \overline{z} \). Determine the radius of convergence of the series. Hint: See #4 of Homework 7 from last semester.

5. Find a conformal map \( w(z) \) of the slit plane \( \mathbb{C} - (-\infty, 0] \) onto the open unit disk satisfying \( w(0) = i \), \( w(-1 + 0i) = +1 \), \( w(-1 - 0i) = -1 \). What are the images of circles centers at 0 under the map? Sketch them.