Homework 5, Math 422, Spring 2004

Due Friday, March 19

1. Let \( \{f_n(z)\} \) be a uniformly bounded sequence of holomorphic functions on a domain \( D \), and let \( z_0 \in D \). Suppose that for each \( m \geq 0 \), \( f_n^{(m)}(z_0) \to 0 \) as \( n \to \infty \). Show that \( f_n(z) \to 0 \) normally on \( D \).

2. Let \( D \) be a bounded domain, and let \( f(z) \) be a holomorphic function from \( D \) into \( D \). Denote by \( f_n(z) \) the \( n^{th} \) iterate \( (f \circ f \circ \cdots \circ f)(z) \). Suppose that \( z_0 \) is an attracting fixed point for \( f(z) \), i.e., \( f(z_0) = z_0 \) and \( |f'(z_0)| < 1 \). Show that \( f_n(z) \) converges uniformly on compact subsets to \( z_0 \). (Hint: Reduce to the case \( z_0 = 0 \), \( \mathbb{D} \subset D \), and \( f(\mathbb{D}) \subset \mathbb{D} \). Use the Schwarz Lemma.)

3. Let \( \varphi(z) \) be the Riemann map of a simply connected domain \( D \) onto \( \mathbb{D} \), normalized by \( \varphi(z_0) = 0 \) and \( \varphi'(z_0) > 0 \) (real). Show that if \( f(z) \) is any holomorphic function on \( D \) with \( f(z_0) = 0 \) and \( |f(z)| \leq 1 \) for all \( z \in D \), then \( |f'(z_0)| \leq \varphi'(z_0) \), with equality only when \( f(z) \) is a constant multiple of \( \varphi(z) \). Remark: This shows that \( \varphi(z) \) is the Ahlfors function of \( D \) corresponding to \( z_0 \).

4. Show that

\[
\left( \frac{1}{f} \right)^{\frac{1}{2}} (z) = f^1(z)
\]

by direct calculation. Also show that

\[
(g \circ f)^1(z) = g^1(f(z))|f'(z)|.
\]

Interpret the latter in terms of stretching with respect to the Euclidean and spherical metrics.

5. Show that the functions \( 1/(z + \epsilon) \), \( 0 < \epsilon \leq 1 \), form a normal family of meromorphic functions on \( (\mathbb{C} \cup \{\infty\}) - \{0\} \). (Hint: Marty’s Theorem.)