Homework 6, Math 422, Spring 2004

Due Friday, April 9

1. Show that the meromorphic function $e^{1/z} + e^{-1/z}$ on $\mathbb{C} - \{0\}$ does not omit any value at $z = 0$.

2. Show that any holomorphic function $f(z)$ on a domain $D$ can be approximated normally on $D$ by a sequence of rational functions that are holomorphic on $D$.

3. Let $\{z_j\}$ be a sequence of distinct points in a domain $D$ that accumulates on $\partial D$, and let $\{w_j\}$ be a sequence of complex numbers. Show that there is a holomorphic function $f(z)$ on $D$ such that $f(z_j) = w_j$ for all $j$. Remark: The sequence $\{z_j\}$ is called an interpolating sequence for holomorphic functions on $D$.

4. Construct a meromorphic function on $\mathbb{C}$ whose poles are simple poles at the Gaussian integers $m + ni$ with residue 1.

5. Use the partial fractions expansion

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{\infty} \frac{1}{(z - k)^2}$$

to show that:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(b) $\pi \cot(\pi z) = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - n^2}$