Calculus II for the Life, Social and Managerial Sciences

Math 128 — Fall 2007

In-term exam November 14

Name: **Student-ID:**

This exam contains sixteen problems. Problems 1 – 14 are multiple choice problems, which each count 5% towards your total score. Problem 15 and 16 will be hand-graded (with a possibility of partial credit) and count 15% each towards your total score.

Problem 1

Which function $y(t)$ is a solution to the following initial value problem?

$$y' - 2y = 0, \quad y(0) = 5$$

$\frac{dy}{dt} = 2y.$

$\int \frac{dy}{y} = \int 2\,dt$

$\ln|y| = 2t + c$

$y = Ae^{2t}$

$y(0) = 5 = A.$

$y(t) = 5e^{2t}.$

A) $y(t) = e^{2t}$

B) $y(t) = e^{5t}$

C) $y(t) = e^{-2t}$

D) $y(t) = e^{-5t}$

E) $y(t) = 5e^{2t}$

F) $y(t) = 2e^{5t}$

G) $y(t) = 5e^{-2t}$

H) $y(t) = 2e^{-5t}$

Problem 2

Find the constant solution of the following differential equation.

$$y' = 3t^2(y+2)^2$$

Constant Solution $\Rightarrow$ $y' = 0$.

$3t^2(y+2)^2 = 0$.

$y+2=0$

$y=-2$.

A) $y = -2$

B) $y = -\sqrt{2}$

C) $y = -1$

D) $y = 0$

E) $y = 1$

F) $y = \sqrt{2}$

G) $y = 2$

H) $y = 3$

Problem 3

Find all other solutions to the differential equation in Problem 2 by actually solving the equation.

A) $y(t) = -\frac{1}{t^3} + C$

B) $y(t) = -\frac{1}{t^3+C} + 2$

C) $y(t) = \frac{1}{t^3} + C$

D) $y(t) = \frac{1}{t^3+C} - 2$

E) $y(t) = \frac{2}{t^3} + C$

F) $y(t) = \frac{1}{t^3+C} + 2$

G) $y(t) = -\frac{2}{t^3} + C$

H) $y(t) = -\frac{1}{t^3+C} - 2$

$\frac{dy}{dt} = 3t^2 (y+2)^2$

$\int \frac{dy}{(y+2)^2} = \int 3t^2 \, dt$

$-\frac{1}{y+2} = t^3 + C.$

$y+2 = -\frac{1}{t^3+C}.$

$y = -2 - \frac{1}{t^3+C}$

Problem 4

Solve the initial value problem

$$y' + 2y = e^{-t}, \quad y(0) = 4$$

$A(t) = \int 2\,dt = 2t$

$e^{A(t)} = e^{2t}$

$e^{2t}y' + 2e^{2t}y = e^{2t}e^{-t}$

$(e^{2t}y)' = e^{t}$

$e^{2t}y = e^{t} + C.$

$y = e^{-t} + Ce^{-2t}$

$4 = y(0) = 1 + C$

$C = 3.$

$y = e^{-t} + 3e^{-2t}$

A) $y(t) = e^{-t} + 3e^{-2t}$

B) $y(t) = 3e^{-t} + e^{-2t}$

C) $y(t) = e^{-t} - 3e^{-2t}$

D) $y(t) = 3e^{-t} - e^{-2t}$

E) $y(t) = -e^{-t} + 3e^{-2t}$

F) $y(t) = -3e^{-t} + e^{-2t}$

G) $y(t) = -e^{-t} - 3e^{-2t}$

H) $y(t) = -3e^{-t} - e^{-2t}$

Problem 5

An initial deposit of \$1,000 is made into an account earning 8% on a yearly basis, compounded continuously. Money is then continuously withdrawn at a constant rate of \$100 a year, until the account is depleted.
Which initial value problem models this situation?

A) $y' = 0.8y + 100, \quad y(0) = 1000$

B) $y' = 0.8y + 1000, \quad y(0) = 100$

C) $y' = 0.08y + 100, \quad y(0) = 1000$

D) $y' = -0.08y + 100, \quad y(0) = 1000$

E) $y' = 0.08y - 100, \quad y(0) = 1000$

F) $y' = 0.08y - 1000, \quad y(0) = 100$

G) $y' = 0.8y - 100, \quad y(0) = 0$

H) $y' = 0.8y - 100, \quad y(0) = 1000$

$y' = \underbrace{.08y}_{\text{interest}} - \underbrace{100}_{\text{withdrawals}}$

$\underbrace{y(0) = 1000}_{\text{initial}}$

Problem 6

Suppose that a fish population in a lake develops according to the logistic equation

$$N'(t) = 0.1N - 0.0001N^2$$

where t is measured in weeks and $N(0) = 100$.
Determine the carrying capacity (K), the intrinsic rate of growth (r) and the size of the population when $N'(t)$ reaches a maximum value (let's call this size M).

A) $K = 1000$, $r = 0.1$, $M = 500$

B) $K = 1000$, $r = 0.1$, $M = 100$

C) $K = 1000$, $r = 0.0001$, $M = 500$

D) $K = 1000$, $r = 0.0001$, $M = 100$

E) $K = 2000$, $r = 0.1$, $M = 500$

F) $K = 2000$, $r = 0.1$, $M = 100$

G) $K = 2000$, $r = 0.0001$, $M = 500$

H) $K = 2000$, $r = 0.0001$, $M = 100$

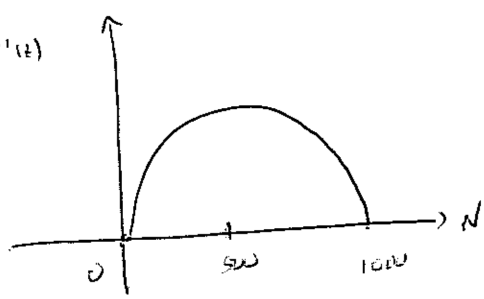
$N'(t) = N(.0001)(1000 - N).$

$= \frac{1}{10000} N (1000 - N).$

$K = 1000$

$\frac{r}{K} = \frac{1}{10000}$, so $r = \frac{1000}{10,000} = 0.1.$

$z = N'(t)$



So $M = 500$

Problem 7

Use Euler's method with $n = 2$ on the interval $2 \leq t \leq 3$ to approximate the solution $y(t)$ to the initial value problem $y' = t - 2y,\ y(2) = 3$. Give as your answer the approximation of $y(3)$.

A) $y(3) \approx 1.05$

B) $y(3) \approx 1.15$

C) $y(3) \approx 1.25$

D) $y(3) \approx 1.35$

E) $y(3) \approx 1.45$

F) $y(3) \approx 1.55$

G) $y(3) \approx 1.65$

H) $y(3) \approx 1.75$

$n=2$, so step size is $\frac{1}{2} = \frac{3-2}{2} = \frac{\text{interval length}}{n}$.

$y(2)=3$, ~~[illegible]~~ slope of tangent line is:

$t-2y = 2 - 2\cdot 3 = 2-6 = -4,$

at $(2,3)$

tangent line at $(2,3)$: $y-3 = -4(x-2)$

$y = -4x+8+3$

$= -4x+11$

$y(2.5) \approx -4(2.5)+11.$

$= -10+11 = 1.$

at 2.5, the slope of the tangent line is: $2.5 - 2(1) = \frac{1}{2}.$

tangent line at $(2.5, 1)$ is: $y - 1 = \frac{1}{2}(x - \frac{5}{2})$

$y = 1 + \frac{1}{2}(x - 5/2)$

so $y(3) \approx 1 + \frac{1}{2}(3-2.5) = 1+\frac{1}{4} = 1.25$

Problem 8

Evaluate

$$\int_0^1 \frac{2x}{(2+x^2)^2}\,dx$$

$u = x^2+2,$

$du = 2x\,dx$

A) $\frac{1}{12}$

B) $\frac{1}{6}$

C) $\frac{1}{4}$

D) $\frac{1}{3}$

E) $\frac{1}{2}$

F) $\frac{2}{3}$

G) $\frac{3}{4}$

H) $\frac{5}{6}$

$$= \int_2^3 \frac{du}{u^2} = -\frac{1}{u}\Big|_2^3 = -\left[\frac{1}{3}-\frac{1}{2}\right].$$

$$= \frac{1}{6}.$$

Problem 9

Find the first Taylor polynomial of $f(x) = \sqrt{x}$ at $x = 2$, and use it to approximate $\sqrt{2.3}$.

$f(x) = \sqrt{x}$, $f(2) = \sqrt{2}$.

$f'(x) = \frac{1}{2\sqrt{x}}$, $f'(2) = \frac{1}{2\sqrt{2}}$.

A) 1.5103

B) 1.5123

C) 1.5143

D) 1.5163

E) 1.5183

F) 1.5203

G) 1.5223

H) 1.5243

first degree Taylor poly.

$\sqrt{2} + \frac{1}{2\sqrt{2}}(x-2)$

$\sqrt{2.5} \approx f(2.3) \approx \sqrt{2} + \frac{1}{2\sqrt{2}}(2.3-2)$.

≈ 1.5203

Problem 10

Compute the second Taylor polynomial of $f(x) = e^{x^2}$ at $x = 0$, and use it to approximate the area under the graph of f from $x = 0$ to $x = 1$.

$$\int_0^1 e^{x^2}\, dx \approx \cdots$$

$$e^{x^2} \approx \underbrace{1 + \quad x^2}_{\text{2nd degree Taylor polynomial.}}$$

$$\int_0^1 e^{x^2}\, dx \approx \int_0^1 1+x^2 = x + \frac{x^3}{3}\Big|_0^1$$

$$= 1 + \frac{1}{3} = 4/3$$

A) $\frac{3}{5}$

B) $\frac{2}{3}$

C) $\frac{3}{4}$

D) $\frac{4}{5}$

E) $\frac{5}{4}$

F) $\frac{4}{3}$

G) $\frac{3}{2}$

H) $\frac{5}{3}$

Problem 11

Find the third Taylor polynomial of $f(x) = x^4$ at $x = 2$.

A) $16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3$

B) $16 + 8(x-2) + 32(x-2)^2 + 24(x-2)^3$

C) $16 + 24(x-2) + 8(x-2)^2 + 32(x-2)^3$

D) $16 + 32(x-2) + 8(x-2)^2 + 24(x-2)^3$

E) $16 + 24(x-2) + 32(x-2)^2 + 8(x-2)^3$

F) $16 + 8(x-2) + 24(x-2)^2 + 32(x-2)^3$

G) $32 + 16(x-2) + 8(x-2)^2 + 24(x-2)^3$

H) $32 + 24(x-2) + 16(x-2)^2 + 8(x-2)^3$

$f(x) = x^4 \quad f(2) = 16$

$f'(x) = 4x^3, \quad f'(2) = 32$

$f''(x) = 12x^2, \quad f''(2) = 48$

$f^{(3)}(x) = 24x, \quad f^{(3)}(2) = 48$

$$P_3(x) = 16 + 32(x-2) + \frac{48}{2}(x-2)^2 + \frac{48}{3!}(x-2)^3.$$

$$= 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3$$

Problem 12

Evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(\pi - 2x)\, dx$$

$u = \pi - 2x$

$du = -2dx$

A) 0

B) $\frac{1}{8}$

C) $\frac{1}{6}$

D) $\frac{1}{4}$

E) $\frac{1}{3}$

F) $\frac{1}{2}$

G) 1

H) 2

$= -\frac{1}{2}\int_{\frac{\pi}{3}}^{0} \sin(u)\, du$

$= \frac{1}{2}\left[\cos(u)\right|_{\pi/3}^{0} = \frac{1}{2}\left[1 - \frac{1}{2}\right] = \frac{1}{4}$

Problem 13

Find the sum of the geometric series $5 + 4 + 3.2 + 2.56 + 2.048 + \cdots$, if it exists.

A) 5

B) 10

C) 15

D) 20

E) 25

F) 30

G) 35

H) This series diverges.

$$= 5(1 + .8 + (.8)^2 + (.8)^3 + \cdots).$$

$a = 5$

$r = .8. = 4/5,$

$$= \frac{5}{1-\frac{4}{5}} = \frac{5}{\frac{1}{5}} = 25$$

Problem 14

What is the Taylor series for $f(x) = 2x(\frac{1}{1-x} + 1)$?

A) $2x^2 + 2x^3 + 2x^4 + 2x^5 + \cdots$

B) $4x + 2x^2 + 2x^3 + 2x^4 + \cdots$

C) $2x + 2x^2 + 2x^3 + 2x^4 + \cdots$

D) $4 + 2x + 2x^2 + 2x^3 + \cdots$

E) $2x + 2x^2 + 2x^3 + 2x^4 + \cdots$

F) $2x + 4x^2 + 4x^3 + 4x^4 + \cdots$

G) $4x + 2x^2 + 4x^3 + 2x^4 + \cdots$

H) $2x + 4x^2 + 2x^3 + 4x^4 + \cdots$

Taylor Series for $\frac{1}{1-x} = 1 + x + x^2 + \cdots$

for $\frac{2x}{1-x} = 2x + 2x^2 + 2x^3 + \cdots$

$f(x) = \frac{2x}{1-x} + 2x$

has taylor Series $4x + 2x^2 + 2x^3 + 2x^4 + \cdots$

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Written problem - Show your work

Problem 15

Solve the initial value problem

$$y' - 2ty = -2t, \quad y(0) = 7$$

$y' = 2ty - 2t = 2t(y-1)$.

$\int \frac{dy}{y-1} = \int 2t \, dt$

$\ln|y-1| = t^2 + C$.

$y - 1 = Ae^{t^2}$, A any real Number

$y = 1 + Ae^{t^2}$

$7 = y(0) = 1 + A$

$A = 6$.

$y = 1 + 6e^{t^2}$

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Written problem - Show your work

Problem 16

Find the third Taylor polynomial of $f(x) = \sin(3x)$ at $x = 0$ (call it $p_3(x)$).

$f(x) = \sin(3x)$ $\quad f(0) = 0$

$f'(x) = 3\cos(3x)$, $\quad f'(0) = 3$

$f''(x) = -9\sin(3x)$, $\quad f''(0) = 0$

$f^{(3)}(x) = -27\cos(3x)$, $\quad f^{(3)}(0) = -27$.

$p_3(x) = 0 + 3x + 0 + -\frac{27}{3!}x^3$.

$= 3x - \frac{27}{3!}x^3$.