

EXAM 2, MATH 128
MONDAY, MARCH 14, 2005

This examination has 18 multiple choice questions, and two essay questions. Please check it over and if you find it to be incomplete, notify the proctor. Do all your supporting calculations in this booklet. In case of a doubtful mark on your answer card, your booklet can then be checked. When you mark your card, use a soft lead pencil (#2). Erase fully any answers you want to change. Problems 1 through 18 are worth 4 points apiece for a total of 72 points.

On problems 19 and 20, show all your work and indicate clearly your answer to the problem. Partial credit will be given for partially completed solutions. Each of these problems is worth 14 points. A good test strategy is to start with these two problems, work on them for no more than a total of 40 minutes, then work rapidly on the first 18 questions and do as many of them as you can in the time that remains.

You may use a 3 x 5 note card and any scientific calculator other than a CAS calculator. Thus, among Texas Instrument calculators, the TI-83, TI-83+, TI-85, and TI-86 are all fine but the TI-89 and TI-92 ARE NOT ALLOWED.

(1) Evaluate $f_y = \partial f / \partial y$ for the function $f(x, y) = \ln(2x^2y)$.

- (A) 0
- (B) $\frac{1}{2x^2}$
- (C) $\frac{1}{2x^2y}$
- (D) $\frac{y}{2x^2}$
- (E) $\frac{1}{y}$
- (F) $\frac{2x^2}{y}$
- (G) $\frac{1}{2x^2} + \frac{1}{y}$
- (H) y
- (I) $2x^2$
- (J) $2x^2y$

$$f_y = \frac{1}{2x^2y} \cdot 2x^2 = \frac{1}{y}$$

(2) Evaluate $f_{xy} = \partial/\partial y(\partial f/\partial x)$ for the function $f(x, y) = x^2y^4 + e^{xy} + 5$.

- (A) $y^2 + e^{xy}$
 (B) $8y^3 + e^{xy}$
 (C) $4xy^3 + ye^{xy}$
 (D) $2xy^3 + xe^{xy}$
 (E) $xy^3 + xye^{xy}$
 (F) $4xy^3 + x^2y^2e^{xy}$
 (G) $8y^3 + x^2e^{xy}$
 (H) $2x + 4y^3 + 5$
 (I) $2x + 4y^3 + xye^{xy}$
 (J) $8xy^3 + xye^{xy} + e^{xy}$

$$f_x = 2xy^4 + ye^{xy}$$

$$f_{xy} = 8xy^3 + e^{xy} + xye^{xy}$$

(3) Evaluate $f_{xx} + f_{yy}$ for $f(x, y) = \ln(x^2 + y^2)$

- (A) 0
 (B) $1/x^2 + 1/y^2$
 (C) $-2/x^3 + 2/y^3$
 (D) $4xy/(x^2 + y^2)^2$
 (E) $4/(x^2 + y^2)^2$
 (F) $4/(x^2 + y^2)$
 (G) $4xy/(x^2 + y^2)$
 (H) $2\ln(x^2 + y^2)$
 (I) $(x^2 + y^2)\ln(x^2 + y^2)$
 (J) $4xy\ln(x^2 + y^2)$

$$f_x = \frac{1}{x^2+y^2} \cdot 2x$$

$$f_{xx} = \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2}$$

$$= \frac{2(y^2 - x^2)}{(x^2+y^2)^2}$$

$$f_{yy} = \frac{2(x^2 - y^2)}{(x^2+y^2)^2}$$

$$f_{xx} + f_{yy} = 0$$

- (4) Find the maximum value of the function $f(x, y) = xy e^{-xy}$; using theorems on continuous functions, a maximum value for f is guaranteed to exist.

- (A) 0
 (B) 1
 (C) e^{-1}
 (D) e^{-2}
 (E) e^{-4}
 (F) $2e^{-2}$
 (G) $4e^{-4}$
 (H) e
 (I) $4e^2$
 (J) $4e^4$

$$f_x = ye^{-xy} - xy^2 e^{-xy} \quad \left| \quad f_y = xe^{-xy} - yx^2 e^{-xy}\right.$$

$$e^{-xy} y(1 - xy) = 0 \quad \left| \quad xe^{-xy}(1 - xy) = 0\right.$$

occurs when $y=0$ OR $xy=1$ occurs when $x=0$
OR $xy=1$.

Critical points $(0,0)$ OR points $(x, \frac{1}{x})$.

Max must occur at a critical point,

So, either it's $f(0,0)=0$ OR:

Since $xy=1$ at other critical points, at these points,

$$f(x, y) = xy e^{-xy} = 1 \cdot e^{-1}$$

Maximum Value is e^{-1}

- (5) The function $f(x, y) = x^3/3 + y^3/3 - 6xy + 15$ has $(0,0)$ and $(6,6)$ as critical points. Which of the following describes the behaviour of f near these two points?

- (A) Both are saddle points for f .
 (B) Both are local minimum points for f .
 (C) Both are local maximum points for f .
 (D) These points are neither local max/min nor saddle points for f .
 (E) $(0,0)$ is a local minimum point for f , $(6,6)$ is a saddle point.
 (F) $(0,0)$ is a saddle point for f , $(6,6)$ is a local minimum point.
 (G) $(0,0)$ is a local maximum point for f , $(6,6)$ is a saddle point.
 (H) $(0,0)$ is a saddle point for f , $(6,6)$ is a local minimum point.
 (I) $(0,0)$ is a local minimum point for f , $(6,6)$ is a local maximum point.
 (J) $(0,0)$ is a local maximum point for f , $(6,6)$ is a local minimum point.

Do 2nd Derivative Test:

$$f_x = x^2 - 6y$$

$$f_{xy} = -6$$

$$f_y = y^2 - 6x$$

$$f_{xx} = 2x$$

$$f_{yy} = 2y$$

At each point, $AC - B^2 = 4xy - 36$

at $(0,0)$ This is -36 , so $(0,0)$ a Saddle

at $(6,6)$ $AC - B^2 = 36 \cdot 3 > 0$, $A = 12 > 0$, $(6,6)$ a Min.

(6) For the function $f(x, y) = x^2 - x^2y^2 + y^2$, how many critical points does f have?

- (A) none
 (B) 1
 (C) 2
 (D) 3
 (E) 4
 (F) 5
 (G) 6
 (H) 7
 (I) 8
 (J) 9

$$\begin{aligned} f_x = 2x - 2xy^2 = 0 & \quad \left| \quad f_y = -2x^2y + 2y = 0 \\ 2x(1-y^2) = 0 & \quad \left| \quad 2y(1-x^2) = 0 \\ x=0, \text{ or } y = \pm 1 & \quad \left| \quad y=0 \text{ or } x = \pm 1 \end{aligned}$$

Critical points: $(0,0)$ $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$

5 of them.

(7) Find the maximum value of $f(x, y) = 6xy - 5$ subject to the constraint $g(x, y) = 2$ where $g(x, y) = x^2 + y^2$.

- (A) -5
 (B) -4
 (C) -3
 (D) -2
 (E) -1
 (F) 0
 (G) 1
 (H) 2
 (I) 3
 (J) 4

Step 1: $g_x = 2x = 0$ when $x=0$, $g_y = 2y = 0$ when $y=0$.
 $g(0,0) \neq 2$.

Step 2: No endpoints

Step 3: $f_x = 6y$, $f_y = 6x$

$f_x g_y = f_y g_x$ and $x^2 + y^2 = 2$

$(6y)(2y) = (6x)(2x)$

$x^2 = y^2$

then $\sqrt{x^2 + x^2 = 2}$

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$

$y = \pm 1$

(x,y) Points	$f(x,y)$
$(1,1)$	1
$(1,-1)$	-11
$(-1,1)$	-11
$(-1,-1)$	1

- (8) For a certain company, units of labor and capital have been chosen so that the cost of each unit of labor utilized is \$1 million and the cost of each unit of capital utilized is \$4 million. The productivity function for this company is given in millions of produced goods by $P(x, y) = 10\sqrt{xy}$ where x is the number of units of labor utilized and y is the number of units of capital utilized. If the company is constrained to operate with a fixed total cost of \$16 million for labor and capital, what is the maximum value for $P(x, y)$ which can be achieved?

- (A) 10
- (B) 20
- (C) 30
- (D) 40
- (E) 50
- (F) 60
- (G) 70
- (H) 80
- (I) 90
- (J) 100

units for \$ are millions,

Constraint: $g(x, y) = x + 4y = 16$.

$P(x, y) = 10x^{1/2}y^{1/2}$

Step 1: $g_x = 1, g_y = 4$. No points.

Step 2: Endpoints: $x=0, y=4$ and $x=16, y=0$. If y or $x = 0, P = 0$. So the shouldn't occur.

Step 3: $P_x = 5x^{-1/2}y^{1/2}$

$P_y = 5x^{1/2}y^{-1/2}$

$P_x = P_y$ is:

(4) $5x^{-1/2}y^{1/2} = 5x^{1/2}y^{-1/2}$

$ty = x$

and $x + 4y = 16$.

$2x = 16$

$x = 8$

$y = 2$.

$P(8, 2) = 10\sqrt{16} = 40$.

- (9) Find the minimum value of $f(x, y) = 3x + 2y$ subject to the constraints $2x + y = 1$ and both x and $y \geq 0$. Method 1:

- (A) 1
- (B) 3/2
- (C) 2
- (D) 5/2
- (E) 3
- (F) 7/2
- (G) 4
- (H) 9/2
- (I) 5
- (J) 11/2

$y = 2 + y$

Step 1: $g_x = 3, g_y = 2$ no points.

Step 2: if $x=0, y=1$ (0,1)
 $y=0, x=1/2$ (1/2, 0)

Step 3: $f_x = 3, f_y = 2$

$3 = 2$ No points.
never.

Method 2

$y = 1 - 2x$, substitute:

$f(x) = 3x + 2(1 - 2x)$
 $= -x + 2$.

This function has no min



So check end points of constraint, proceed as in step 2

(min) points	f(min)
(0, 1)	2
(1/2, 0)	3/2

Min

- (10) Let $y = ax + b$ be the regression line for the collection of six (x, y) data points $(0, 2)$, $(1, 4)$, $(2, 7)$, $(3, 6)$, $(4, 8)$, $(5, 11)$. Using your calculator, find to two decimal places the fraction r^2 of the variance in the y -data explained by the regression equation.

- (A) .83
(B) .84
(C) .85
(D) .86
(E) .87
(F) .88
(G) .89
(H) .90
(I) .91
(J) .92

$$r^2 \approx .908$$

$$\text{Rounded: } .91$$

- (11) Use your calculator to find the quadratic prediction equation for y in terms of x which best fits the five (x, y) data points $(0, 6)$, $(1, 2)$, $(2, 1)$, $(3, 4)$, $(4, 12)$ in the least squares sense. What does this quadratic regression model predict for the value of y when $x = 5$?

- (A) 13.9
(B) 14.6
(C) 15.3
(D) 16.2
(E) 16.7
(F) 17.1
(G) 18.5
(H) 19.8
(I) 21.4
(J) 23.2

$$y = 2x^2 - 6.5x + 6.2$$

$$y(5) = 23.2$$

(12) Evaluate $\int (2x - 2xe^{2y} - 3) dx = x^2 - x^2e^{2y} - 3x + C(y)$

- (A) $x^2 - y^2e^{2y} - 3y + C(y)$
 (B) $x^2 - x^2e^{2y} - 3x + C(y)$
 (C) $x^2 - 2e^{2y} - 3x + C(y)$
 (D) $2x - x^2e^{2y} - 3x^2 + C(y)$
 (E) $x^2 - x^2ey^{2y} - 3xy + C(y)$
 (F) $x^2 - x^2e^{2y}/y - 3x + C(y)$
 (G) $xy - x^2e^{2y} - 3xy + C(y)$
 (H) $x^2y - x^2e^{2y}/y - 3x + C(y)$
 (I) $xy - x^2e^{2y}/y - 3x + C(y)$
 (J) $xy - xe^{2y} - 3xy + C(y)$

- (13) Find the volume under the graph of $z = 20(x + y)^3$ and above the rectangle $[0, a] \times [0, b]$. Hint: Do not multiply out $(x + y)^3$; instead, just do the two iterated integrals by substitution.

- (A) $4(a + b)^3 - 4a^3 - 4b^3$
 (B) $5(a + b)^4 - 5a^4 - 5b^4$
 (C) $3(a + b)^5 - 2a^5 - b^5$
 (D) $2(a + b)^5 + a^5 + b^5$
 (E) $(a + b)^5 - a^5 - b^5$
 (F) $(a + b)^6 - a^6 - b^6$
 (G) $(a + b)^5 + 10(a^2b^3 + a^3b^2)$
 (H) $(a^5 + b^5 - 5(a^4 + b^4))$
 (I) $(a^4 + b^4 - 6a^2b^2)$
 (J) $(5a^4 + 5b^4 - 10(a + b)^3)$

$$\begin{aligned}
 20 \int_0^a \int_0^b (x+y)^3 dy dx &= 20 \int_0^a \left[\frac{(x+y)^4}{4} \right]_0^b dx \\
 &= 5 \int_0^a (b+x)^4 - x^4 dx \\
 &= \frac{1}{5} \left[\frac{(b+x)^5}{5} - \frac{x^5}{5} \right]_0^a \\
 &= (b+a)^5 - a^5 - b^5
 \end{aligned}$$

(14) Find the average value of $C(x, y) = 15(x^2 + y^2)$ over the rectangle $[-1, 1] \times [-2, 2]$.

- (A) 20
 (B) 22.5
 (C) 25
 (D) 27.5
 (E) 30
 (F) 32.5
 (G) 35
 (H) 37.5
 (I) 40
 (J) 42.5

Area of Rectangle is: $4 \times 2 = 8$.

$$15 \int_{-1}^1 \int_{-2}^2 x^2 y^2 dy dx = 15 \int_{-1}^1 \left[x^2 y + \frac{y^3}{3} \right]_{-2}^2 dx$$

$$= 15 \int_{-1}^1 \left[2x^2 + \frac{8}{3} - \left[-2x^2 + \frac{8}{3} \right] \right] dx$$

$$= 15 \int_{-1}^1 \left[4x^2 + \frac{16}{3} \right] dx = 15 \left[\frac{4}{3} x^3 + \frac{16}{3} x \right]_{-1}^1$$

$$= 5 \left[\frac{4}{3} + \frac{16}{3} - \left(-\frac{4}{3} + \frac{16}{3} \right) \right]$$

$$= 5 [4 + 16 + 4 + 16] = 200$$

Average Value: $\frac{200}{8} = 25$

(15) What is the general solution of the differential equation $y' = 4y$?

- (A) $y = x^3/3 + C$
 (B) $y = x^2/2 + C$
 (C) $y = x^4 + C$
 (D) $y = 4 \ln x + C$
 (E) $y = e^{4x} + C$
 (F) $y = 2e^{2x} + C$
 (G) $y = Ce^{4x}$
 (H) $y = Ce^{x^2}$
 (I) $y = Ce^{x^2/2}$
 (J) $y = Ce^{x^4}$

$$\frac{dy}{dx} = 4y$$

$$\int \frac{dy}{y} = \int 4 dx$$

$$\ln y = 4x + C$$

$$y = Ce^{4x}$$

(16) Find the general solution of the differential equation $y' = \frac{4x^3}{3y^2}$.

- (A) $y = (C + 4x^4)^3$
 (B) $y = C + 4x^4/3$
 (C) $y = (C + x^4)^3$
 (D) $y = (C + 2x^4)^{2/3}$
 (E) $y = (C + x^4)^{1/3}$
 (F) $y = (C + x^3)^{1/4}$
 (G) $y = (C + 4x^3)^{1/4}$
 (H) $y = (C + 2x^3)^{1/2}$
 (I) $y = (C + x^2)^{4/3}$
 (J) $y = (C + x)^{3/4}$

$$\frac{dy}{dx} = \frac{4x^3}{3y^2}$$

$$\int 3y^2 dy = \int 4x^3 dx$$

$$y^3 = x^4 + C$$

$$y = (x^4 + C)^{1/3}$$

(17) Find the particular solution of the differential equation $y' = \frac{2x}{y}$ for which

$$y(x) > 0 \text{ for all } x \text{ and } y(1) = 2$$

- (A) $y = \sqrt{2x^2 + 2}$
 (B) $y = \sqrt{3x^2 + 1}$
 (C) $y = \sqrt{4x^2 - 4}$
 (D) $y = \sqrt{4x^2 + 2}$
 (E) $y = \sqrt{6x^2 - 2}$
 (F) $y = \sqrt{x^3 + 3}$
 (G) $y = \sqrt{2x^3 + 2}$
 (H) $y = \sqrt{x^4 + 3}$
 (I) $y = x^2 + 1$
 (J) $y = 2x^2$

$$\frac{dy}{dx} = \frac{2x}{y} \Rightarrow \int y dy = \int 2x dx$$

$$\frac{y^2}{2} = x^2 + C, \quad y^2 = 2x^2 + C$$

$$y = \sqrt{2x^2 + C} \quad (\text{since } y > 0)$$

$$2 = y(1) = \sqrt{2 + C}$$

$$4 = 2 + C$$

$$C = 2$$

$$y = \sqrt{2x^2 + 2}$$

(18) Evaluate $\int_R ye^{xy} dA$ for $R = [0, 1] \times [1, 2]$.

(A) $e^2 - 2e + 1$

(B) $e^2 - 2e$

(C) $e^2 - 3$

(D) $e^2 - e - 1$

(E) $e - 2$

(F) $e - 2 + 1/e$

(G) $e^2/2 - e + 1/2$

(H) $e^2/2 - e - 1$

(I) $e^2 - 4e + 1$

(J) $e^4 - e^2 - 1$

$$\int_1^2 \int_0^1 ye^{xy} dx dy$$

$$u = xy \\ du = y dx$$

$$= \int_1^2 [e^{xy}]_{x=0}^1 dy = \int_1^2 e^y - 1 dy$$

$$= e^y - y \Big|_1^2 = e^2 - 2 - (e - 1)$$

$$= e^2 - e - 1$$

N.B. FOR PROBLEMS 19 AND 20, LITTLE OR NO CREDIT WILL BE GIVEN IF YOU DON'T SHOW ALL OF YOUR WORK IN THE SPACES PROVIDED OR, IF YOU NEED MORE SPACE, ON THE BACK SIDE OF THE PAGE. TRY TO WRITE CLEARLY AND LABEL YOUR ANSWERS.

- (19) Suppose an insect colony is centered at $(0, 0)$ and the concentration $C(x, y)$ of insects per square foot at (x, y) is given by

$$C(x, y) = 600 - 3(x^2 + y^2)$$

when $|x|$ and $|y|$ are both < 10 .

- (a) For $|x|, |y| < 10$, find the unique anti-derivative $F(x, y)$ of $C(x, y)$ with respect to x for which $F(0, y) = 0$.

$$F(x, y) = \int (600 - 3(x^2 + y^2)) dx = 600x - x^3 - 3y^2x + G(y) \rightarrow \text{"constant of integration"}$$

$$0 = F(0, y) = G(y) \Rightarrow G(y) = 0,$$

$$\Rightarrow F(x, y) = 600x - x^3 - 3y^2x$$

- (b) Calculate $G(w, h) = \int_0^h \int_0^w C(x, y) dx dy$ for each h and w between 0 and 10.

$$\begin{aligned} \int_0^h \int_0^w (600 - 3(x^2 + y^2)) dx dy &= \int_0^h [600x - x^3 - 3y^2x]_0^w dy \\ &= \int_0^h (600w - w^3 - 3y^2w) dy = (600w - w^3)y - y^3w \Big|_0^h \\ &= (600w - w^3)h - h^3w \end{aligned}$$

- (c) Use your formula in (b) to determine the average number of insects per square foot in the rectangle $[0, 4] \times [0, 8]$.

$$\text{Average is } \frac{1}{32} G(4, 8) = \frac{1}{32}$$

- (d) Explain in words (no need to carry out any more integrals) why the average number of insects per square foot in the rectangle $[-4, 4] \times [-8, 8]$ is exactly the same as the average number in $[0, 4] \times [0, 8]$.

$$\text{Average over } [-4, 4] \times [-8, 8] = \frac{1}{2 \cdot 8 \cdot 2 \cdot 4} 4G(4, 8)$$

$$\text{Since } \int_{-4}^4 \int_{-4}^4 C(x, y) dx dy = 4 \int_0^4 \int_0^4 C(x, y) dx dy = 4G(4, 8) \text{ by Symmetry.}$$

$$\text{Average over } [-4, 4] \times [-8, 8] = \frac{1}{4 \cdot (8 \cdot 4)} \cdot 4G(4, 8) = \frac{G(4, 8)}{8 \cdot 4}$$

$$= \text{Average over } [0, 4] \times [0, 8]$$

(20) Suppose a manufacturing company produces both Brand A calculators and Brand B calculators. Let $C(x, y) = x^2 + y^2$ be the manufacturer's cost in producing x units of Brand A and y units of Brand B.

(a) If shipping considerations dictate that $x + y = 90$, what values of x and y minimize $C(x, y)$?

$g(x, y) = x + y = 90$, $C(x, y) = x^2 + y^2$

Step 1: $g_x = 1$, No point w/ $g_x = 0 = g_y$
AND $g(x, y) = 90$

Step 2: Endpoints: $x=0, y=90$, ~~$x=90, y=0$~~
 $y=0, x=90$

Step 3: $g_y C_x = g_x C_y$ and $x + y = 90$
 $2x = 2y$ so $x = y$, $2x = 90$
 $x = 45, y = 45$

(x, y) Points	$C(x, y)$
$(90, 0)$	8100
$(0, 90)$	8100
$(45, 45)$	4050

$(x, y) = (45, 45)$
Minimizes Cost

(b) Suppose the company resolves its shipping issues so that the productivity $x + y$ is no longer constrained. What is the ratio y/x which maximizes productivity when $C(x, y)$ is constrained to be some constant C_0 ?

Maximize $f(x, y) = x + y$ Subject to $C(x, y) = x^2 + y^2 = C_0$

Step 1: $C_x = 2x = 0$ when $x=0$, $C_y = 0$ when $y=0$,
 ~~$C(0,0) = C_0$~~ only when $C_0 = 0$, absurd case

Step 2: Endpoints: $x, y \geq 0$, $x=0, y=\sqrt{C_0}$,
 $y=0, x=\sqrt{C_0}$

Step 3: Solve: $2x = 2y$ and $x^2 + y^2 = C_0$,
 $x = y$

$2x^2 = C_0$,
 $x = \sqrt{\frac{C_0}{2}}, y = \sqrt{\frac{C_0}{2}}$

(x, y) Points	$x + y$
$(0, \sqrt{C_0})$	$\sqrt{C_0}$
$(\sqrt{C_0}, 0)$	$\sqrt{C_0}$
$(\sqrt{\frac{C_0}{2}}, \sqrt{\frac{C_0}{2}})$	$2\sqrt{\frac{C_0}{2}}$

$\frac{y}{x} = \frac{\sqrt{\frac{C_0}{2}}}{\sqrt{\frac{C_0}{2}}} = 1$ Max

(c) Suppose the company sells its Brand A calculators to distributors at \$50/calculator and sells its Brand B calculators at \$100/calculator. What choice of x and y maximize the company's profit? Hint: profit = revenue - cost.

Revenue: $50x + 100y$ Cost: $x^2 + y^2$

Maximize: $f(x, y) = 50x + 100y - x^2 - y^2$

$f_x = 50 - 2x = 0$ when $x = 25$

$f_y = 100 - 2y = 0$ when $y = 50$

2nd Derivative Test:

$f_{xx} = -2$

$f_{yy} = -2$

$f_{xy} = 0$

$AC - B^2 = 4 > 0$

$A < 0$

So critical point $(25, 50)$ is a Max.