EXAM 3, MATH 128
TUESDAY, APRIL 13, 2005

This examination has 18 multiple choice questions, and two essay questions. Please check it over and if you find it to be incomplete, notify the proctor. Do all your supporting calculations in this booklet. In case of a doubtful mark on your answer card, your booklet can then be checked.. When you mark your card, use a soft lead pencil (#2). Erase fully any answers you want to change. Problems 1 through 18 are worth 4 points apiece for a total of 72 points.

On problems 19 and 20, show all your work and indicate clearly your answer to the problem. Partial credit will be given for partially completed solutions. Each of these problems is worth 14 points. A good test strategy is to start with these two problems, work on them for no more than a total of 40 minutes, then work rapidly on the first 18 questions and do as many of them as you can in the time that remains.

You may use a 3 x 5 note card and any scientific calculator other than a CAS calculuator. Thus, among Texas Instrument calculators, the TI-83, TI-83+, TI-85, and TI-86 are all fine but the TI-89 and TI-92 ARE NOT ALLOWED.

(1) Which of the following is the solution of the differential equation

$$y' = -k(y - 10)) \text{ for which } y(0) = 6?$$

(A) $y = 6 + 4e^{-kt}$
(B) $y = 6 - 8e^{-kt}$
(C) $y = 4 - 10e^{-kt}$
(D) $y = 4 + 6e^{-kt}$
(E) $y = 4 + 10e^{-kt}$
(F) $y = 10 + 4e^{-kt}$
(G) $y = 10 - 4e^{-kt}$
(H) $y = 10 - 8e^{-kt}$
(I) $y = 8 - 10e^{-kt}$
(J) $y = 8 - 4e^{-kt}$

$y' = k(10-y)$

$y = 10 + (y_0 - 10)e^{-kt}$

$y = 10 + (6-10)e^{-kt}$

$y = 10 - 4e^{-kt}$

(2) Which of the following is the general solution of the differential equation

$$y' + 2y/x = 5x^2?$$

(A) $y = x^5 + Cx^{-3}$
(B) $y = x^3 + C$
(C) $y = x^5 + C$
(D) $y = x^2 + Cx^5$
(E) $y = x^3 + Cx^{-2}$
(F) $y = x^5 + Cx^4$
(G) $y = x^5 + Cx^2$
(H) $y = 5x^3/3 + C$
(I) $y = 5x^3/3 + C\ln x$
(J) $y = 5x^3/3 + Cx^{-2}$

$I(x) = \int \frac{2}{x}dx = 2\ln x = \ln x^2$

$e^{I(x)} = x^2$

$x^2y' + x^2\frac{2y}{x} = 5x^4$

$\frac{d}{dx}(x^2y) = 5x^4$

$x^2y = \int 5x^4dx = x^5 + c$

$y = x^3 + \frac{C}{x^2}$

(3) Suppose $y(x)$ is the solution of $y' = k(8 - y)$ for which $y(0) = 2$. If $y(2) = 6$, what is the value of e^{-k}?

(A) $\ln 2$
(B) $\ln 3$
(C) $-\ln 2$
(D) $-\ln 3$
(E) $-1/2\ln 3$
(F) $1/\sqrt{3}$
(G) $1/\sqrt{2}$
(H) $1/\sqrt{6}$
(I) $1/2$
(J) 3

$y = 8 + (y_0 - 8)e^{-kx}$, $y_0 = 2$.

$y = 8 - 6e^{-kx}$

$6 = y(2) = 8 - 6e^{-2k}$

$6e^{-2k} = 2$.

$\sqrt{e^{-2k}} = \sqrt{\frac{1}{3}}$

$e^{-k} = \frac{1}{\sqrt{3}}$

(4) Which of the following is a good approximation to $1/(2-\delta)$ when δ is small?

(A) $1/2 - \delta/4$.
(B) $1/2 + \delta/4 + \delta^2/8$.
(C) $1/2 - \delta/4 + \delta^2/8$.
(D) $1/2 - \delta/4 - \delta^2/16$.
(E) $1/2 + \delta/8 + \delta^2/16$.
(F) $1/2 + \delta/6 - \delta^2/12$.
(G) $1/2 + \delta/6 + \delta^2/12$.
(H) $1/2 - \delta/2 + \delta^2/4$.
(I) $1/2 + \delta + \delta^2$.
(J) $1 + \delta/2 + \delta^2/2$.

$$\frac{1}{2-\delta} = \frac{1}{2}\left(\frac{1}{1-\frac{\delta}{2}}\right)$$

2nd degree Taylor poly. for $\frac{1}{1-t}$ is $1+t+t^2$

$$\frac{1}{2-\delta} \sim \frac{1}{2}\left(1 + \frac{\delta}{2} + \frac{\delta^2}{4}\right)$$

$$= \frac{1}{2} + \frac{\delta}{4} + \frac{\delta^2}{8}$$

(5) Which of the following is the second order Taylor polynomial at 0 for the function e^{-3x}?

Sorry ~ ~~(A)~~ $1 - 3x + 9x^2$
(B) $1 - 3x + 9x^2/2$
(C) $1 + 3x + 9x^2/2$
(D) $1 + 3x + 9x^2$
(E) $1 - 3x/2 + 9x^2/6$
(F) $1 + 3x/2 + 9x^2/4$
(G) $1 - 3x/2 - 9x^2/4$
(H) $1 - 3x/6 + 9x^2/12$
(I) $1 + 3x + 9x^2/2$
(J) $1 + 3x + 9x^2/6$

$$1 - 3x + \frac{9x^2}{2}$$

(6) Which of the following is the third order Taylor polynomial at 2 for the function $\ln(x-1)$? Note that $x - 1 = 1 + (x-2)$.

(A) $(x-2) - (x-2)^2/2 + (x-2)^3/3$
(B) $(x-2) - (x-2)^2 + (x-2)^3$
(C) $(x-2) + (x-2)^2 + (x-2)^3$
(D) $(x-2) - (x-2)^2/3 + (x-2)^3/6$
(E) $(x-2) + (x-2)^2/2 + (x-2)^3/6$
(F) $1 - (x-2) + (x-2)^2/2 - (x-2)^3/3$
(G) $1 - (x-2) + (x-2)^2 - (x-2)^3$
(H) $2(x-2) - 3(x-2)^2 + 4(x-2)^3$
(I) $1 + 2(x-2) + 6(x-2)^2 + 12(x-2)^3$
(J) $1 + (x-2) + (x-2)^2/2 + (x-2)^3/6$

$\ln(x-1) = \ln(1+(x-2))$

for $\ln(1+t)$, it's $t - \frac{t^2}{2} + \frac{t^3}{3}$

so, substituting $t = x-2$,

$(x-2) - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3}$

(7) What is the coefficient of x^n in the Taylor series at 0 for the function $\frac{1}{(1-x)^2}$?

(A) $1/2$
(B) 2
(C) 2^n
(D) $(-1)^n$
(E) $(-2)^{n+1}$
(F) $1/n$
(G) $n+1$
(H) $2/(n+1)$
(I) n^2
(J) $1/n!$

$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$

$= \frac{d}{dx}\left(1 + x + \cdots + x^m + \cdots\right)$

$= 1 + 2x + \cdots + mx^{m-1} + \cdots$

let $n = m-1$, get $1 + 2x + \cdots (n+1)x^n + \cdots$

coefficient of x^n is

$(n+1)$

(8) What is the radius of convergence of the Taylor series at 0 for the function $\frac{1}{1+4x}$?

(A) 8
(B) 4
(C) 3
(D) 2
(E) 1
(F) 3/4
(G) 2/3
(H) 1/2
(I) 1/4
(J) 1/8

Interval of conv. for $\frac{1}{1+t}$ is $|t|<1$

Set $t=4x$, so

$|4x|<1$

$|x|<\frac{1}{4}$

(9) Which of the following is the interval of convergence for the Taylor series at 3 of the function $\frac{2}{1+2(x-3)}$?

(A) $1/2 < x < 3/2$
(B) $1/2 < x < 7/2$
(C) $1/2 < x < 11/2$
(D) $3/2 < x < 5/2$
(E) $3/2 < x < 9/2$
(F) $3/2 < x < 13/2$
(G) $2 < x < 4$
(H) $1 < x < 5$
(I) $0 < x < 6$
(J) $5/2 < x < 7/2$

For $\frac{1}{1+t}$ it's $|t|<1$

In this case setting $t=2(x-3)$,

$|2(x-3)|<1$

$|x-3|<\frac{1}{2}$

$\frac{5}{2}<x<\frac{7}{2}$

(10) Suppose the 2nd order Taylor polynomial at 0 for $f(x) = \ln(1+x)$ is used to approximate $\ln 1.1$. Which of the following estimates the error for this approximation by the formula involving the first omitted term?

(A) 1/1000
(B) 1/2000
(C) 1/3000
(D) 1/4000
(E) 1/5000
(F) 1/6000
(G) 1/7000
(H) 1/8000
(I) 1/9000
(J) 1/10000

$f(x) = \ln(1+x) = \underbrace{x - \frac{x^2}{2}}_{P_2(x)} + \underbrace{\frac{x^3}{3}}_{1^{st}\text{ omitted}} - \cdots$

$\ln(1.1) = f\left(\frac{1}{10}\right), \quad |f(x) - P_2(x)| \leq \left|\frac{x^3}{3}\right|$

$\left|f\left(\frac{1}{10}\right) - P_2\left(\frac{1}{10}\right)\right| \leq \left|\left(\frac{1}{10}\right)^3 \frac{1}{3}\right| = \frac{1}{3000}$

(11) Use Taylor polynomials for $f(x) = (100+x)^{1/2}$ to approximate $(101)^{1/2}$ to within $\pm .005$.

(A) 10.100
(B) 10.079
(C) 10.061
(D) 10.057
(E) 10.033
(F) 10.035
(G) 10.038
(H) 10.040
(I) 10.043
(J) 10.050

$(101)^{1/2} = f(1)$.

$f(x) = (100+x)^{1/2} = 10\left(1 + \frac{x}{100}\right)^{1/2}$

$= 10\left(1 + \frac{1}{2}\left(\frac{x}{100}\right) - \frac{1}{4}\left(\frac{x}{100}\right)^2 + \frac{3}{8}\left(\frac{x}{100}\right)^3 - \cdots\right)$

Since we want $f(1)$, and error term $\leq$ 1st omitted term, first term ~~[illegible]~~ that ~~[illegible]~~ is $< \pm .005$ is $\frac{10}{4}\left(\frac{x}{100}\right)^2$ when using $x = 1$.

So, our approximation is

$10\left(1 + \frac{1}{2}\left(\frac{1}{100}\right)\right) = 10.05$

(12) Use the Taylor series at 0 for $f(x) = 4x^5/(1-x)$ to find $f^{(100)}(0)$. Hint: Use the formula for the coefficient a_n of x^n in any Taylor series.

(A) $2 \times 95!$
(B) $4 \times 95!$
(C) $8 \times 95!$
(D) $1/95!$
(E) $2/95!$
(F) $4/95!$
(G) $8/95!$
(H) $1/100!$
(I) $2 \times 100!$
(J) $4 \times 100!$

$$\frac{4x^5}{1-x} = 4x^5(1 + x + x^2 + \cdots + x^n + \cdots)$$

$$= 4x^5 + 4x^6 + \cdots + 4x^{n+5} + \cdots$$

Remember: $\cancel{a_n = \frac{f^{(100)}(0)}{100!}}$ $\quad a_n = \frac{f^{(n)}(0)}{100!}$

So, find a_{100} = coefficient of x^{100}, which is given by 4, since when $n = 95$, $n+5 = 100$.

$$a_{100} = 4 = \frac{f^{(100)}(0)}{100!} \quad \text{so,} \quad f^{(100)}(0) = 4 \times 100!$$

(13) Which of the following expressions arises by using the fourth order Taylor polynomial at 0 for $f(x) = e^{-.1x^2}$ to approximate $\int_0^1 f(x)\,dx$?

(A) $1 + 1/(3 \times 10) + 1/(5 \times 2 \times 100)$
(B) $1 + 1/(3 \times 10) - 1/(5 \times 2 \times 100)$
(C) $1 - 1/(3 \times 10) + 1/(5 \times 100)$
(D) $1 + 1/(3 \times 10) - 1/(5 \times 100)$
(E) $1 - 1/(3 \times 10) + 1/(5 \times 2 \times 100)$
(F) $1 - 1/(3 \times 100) + 1/(5 \times 2 \times 10000)$
(G) $1 + 1/(3 \times 100) - 1/(5 \times 2 \times 1000)$
(H) $1 - 1/10 + 1/(2 \times 100)$
(I) $1 + 1/(2 \times 10) + 1/(3 \times 100)$
(J) $1 + 1/10 + 1/(2 \times 100)$

$$f(x) = e^{-\frac{1}{10}x^2} = \underbrace{1 - \frac{1}{10}x^2 + \frac{1}{2 \times 100}x^4}_{P_4(x)} + \cdots$$

$$\int_0^1 P_4(x)\,dx = \int_0^1 1 - \frac{1}{10}x^2 + \frac{1}{2 \times 100}x^4\,dx$$

$$= x - \frac{x^3}{3 \times 10} + \frac{x^5}{2 \times 5 \times 100}\Big|_0^1$$

$$= 1 - \frac{1}{3 \times 10} + \frac{1}{2 \times 5 \times 100}$$

(14) For $f(x)$ as in the previous problem, supposed we wanted to approximate $\int_0^1 f(x)\,dx$ to within $\pm.0001 = 10^{-4}$. What is the smallest value of n for which the integral of the n^{th} Taylor polynomial at 0 for $f(x)$ will suffice?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5
(F) 6
(G) 7
(H) 8
(I) 9
(J) 10

Taylor Series Alternating, error smaller than absolute value of 1st omitted term. The (2n)th term is given by

$\frac{(-1)^n x^{2n}}{10^n n!}$, Want n such that

$$\underbrace{\int_0^1 \frac{x^{2n}}{10^n n!}\,dx}_{\frac{x^{2n+1}}{10^n n!(2n+1)}\Big|_0^1 = \frac{1}{10^n n!(2n+1)}} < 10^{-4}$$

, Observe if $n=2$ we get $\frac{1}{10^3}$ not good enough, $n=3$, works: $\frac{1}{10^3 3!(7)} < \frac{1}{10^4}$

So, we want error term, 1st omitted term, to be 6 power of x, so polynomial should be of degree 4,

(15) Which of the following is the coefficient of x^6 in the Taylor series at 0 for the function $f(x) = e^{-x^3/2}$?

(A) 1/2
(B) 1/3
(C) 1/4
(D) 1/5
(E) 1/6
(F) 1/7
(G) 1/8
(H) 1/9
(I) 1/10
(J) 1/12

$$1 + \left(\frac{-x^3}{2}\right) + \underbrace{\frac{1}{2!}\left(\frac{-x^3}{2}\right)^2}_{\frac{x^6}{8}} + \cdots$$

(16) For $f(x) = x^{10}/(1-x)^{7/3}$, what is the smallest value of n for which the coefficient of x^n in the Taylor series of f at 0 is not zero?

(A) 0
(B) 1
(C) 5
(D) 10
(E) 11
(F) 15
(G) 20
(H) 21
(I) 25
(J) 30

Taylor series for $\frac{1}{(1-x)^{7/3}}$ around zero starts with

$(1 + \cdots)$

So Taylor series for $f(x)$ is

$x^{10}(1+\cdots) = x^{10} + \cdots$

$n = 10$

(17) The rate of production (in billions of cubic feet per year) of a certain natural gas well is given by

$$R(t) = 3e^{-t/5} - 3e^{-2t/5}.$$

Assuming the well is operated indefinitely, the total production of the well (in billions of cubic feet) is expressed by the integral $\int_0^\infty R(t)\,dt$. What is the value of this integral?

(A) 7.5
(B) 8.0
(C) 8.5
(D) 9.0
(E) 9.5
(F) 10.0
(G) 10.5
(H) 11.0
(I) 11.5
(J) 12.0

$$\lim_{b\to\infty} \int_0^b 3e^{-t/5} - 3e^{-2t/5}\,dt$$

$$= \lim_{b\to\infty} 3\left[-5e^{-t/5} + \frac{5}{2}e^{-2t/5}\right]_0^b$$

$$= \lim_{b\to\infty} 3\left[-5e^{-b/5} + \frac{5}{2}e^{-2b/5} - \left[-5 + 5/2\right]\right]$$

$$= 3\left[\frac{5}{2}\right] = 15/2 = 7.5$$

(18) Let $f(x) = 1/x^4$ for $x \geq 1$ and $f(x) = 0$ for $x < 1$. What is the value of k for which kf is a probability density function?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5
(F) 6
(G) 7
(H) 8
(I) 9
(J) 10

Want $1 = k\int_1^\infty \frac{1}{x^4}\,dx$

Since we need $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$1 = k \lim_{b\to\infty}\int_1^b x^{-4}\,dx = \lim_{b\to\infty} k\left[-\frac{1}{3}x^{-3}\right]_1^b$$

$$= \lim_{b\to\infty} k\left[-\frac{1}{3}\frac{1}{b^3} + \frac{1}{3}\right] = \frac{k}{3}$$

So, $k=3$.

N.B. FOR PROBLEMS 19 AND 20, LITTLE OR NO CREDIT WILL BE GIVEN IF YOU DON'T SHOW ALL OF YOUR WORK IN THE SPACES PROVIDED OR, IF YOU NEED MORE SPACE, ON THE BACK SIDE OF THE PAGE. TRY TO WRITE CLEARLY AND LABEL YOUR ANSWERS.

(19) Let $A(t)$ be the area in square centimeters of a skin wound t days after the wound is incurred. Suppose $A(0) = 15$ and that the healing rate is described by $A'(t) = \frac{-100}{t^2+100}$.

(a) Write down the second order Taylor polynomial $p_2(t)$ at 0 for the function $A'(t)$.

$$\frac{-100}{t^2+100} = \frac{-100}{100}\left(\frac{1}{1+\left(\frac{t}{10}\right)^2}\right) = -\left(1 - \left(\frac{t}{10}\right)^2 + \left(\frac{t}{10}\right)^4 - \cdots + (-1)^n\left(\frac{t}{10}\right)^{2n} + \cdots\right)$$

$$p_2(t) = -1 + \frac{t^2}{100}$$

(b) Use $p_2(t)$ to approximate $A(3)$. Note that $A(3)$ will involve a certain integral of $A'(t)$.

$$A(3) - A(0) \sim \int_0^3 p_2(t)\,dt = \int_0^3 -1 + \frac{t^2}{100}\,dt = -t + \frac{t^3}{300}\Big|_0^3$$

$$= -3 + \frac{9}{100}$$

$$A(3) \approx 15 - 3 + \frac{9}{100} = 12 + \frac{9}{100}$$

(c) An exact calculation of $A(3)$ would entail integrating all of the terms in the Taylor series for $A'(t)$. Use the first omitted term to estimate the error in your approximation of $A(3)$ in part (b).

1st omitted term $-\frac{t^4}{10^4}$

Error between A' and p^2 $< \left|\frac{t^4}{10^4}\right|$

if $A(3)$ = exact value, $\tilde{A}(3)$ = approximate value,

$$|A(3) - \tilde{A}(3)| < \left|A(0) + \int_0^3 A'(t)\,dt - A(0) - \int_0^3 p_2(t)\,dt\right|$$

$$\leq \int_0^3 |A'(t) - p_2(t)|\,dt < \int_0^3 \frac{t^4}{10^4}\,dt$$

$$= \frac{1}{5 \cdot 10^4}\left[t^5\right]_0^3 = \frac{3^5}{5 \cdot 10^4}$$

(20) Let $f(x) = ce^{-4x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

(a) For what value of c is f a probability density function? Briefly explain your answer.

Want $1 = c\int_0^\infty e^{-4x}\,dx = c \lim_{b\to\infty} -\frac{1}{4}e^{-4x}\Big|_0^b = c\left[\frac{1}{4}\right]$

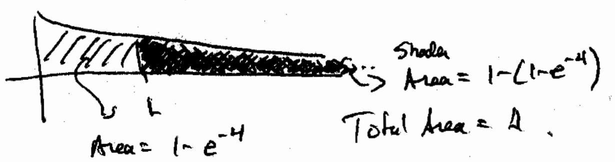
$c = 4$

(b) For the random variable having the probability density function in part (a), what is $P(0 \leq x \leq< 1)$? How can you rapidly obtain $P(x > 1$ from your answer without carrying out another integral calculation?

$\int_0^1 4e^{-4x}\,dx = -e^{-4x}\Big|_0^1 = -\left[e^{-4} - 1\right] = 1 - e^{-4}$

$P(X > 1) = 1 - P(0 \leq X \leq 1) = 1 - (1 - e^{-4}) = e^{-4}$

Since:



(c) For the random variable in part (b), find the number m for which $P(x \leq m) = 1/2$. [m is called the median value of the random variable.]

Solve $\frac{1}{2} = \underbrace{\int_0^m 4e^{-4x}\,dx}_{P(x \leq m)} = -e^{-4x}\Big|_0^m = -e^{-4m} + 1$

$e^{-4m} = \frac{1}{2}$

$-4m = \ln\frac{1}{2} = -\ln 2$

$m = \frac{\ln 2}{4}$