FINAL EXAM , MATH 128
THURSDAY, MAY 5, 2005

This examination has 25 multiple choice questions. Please check it over and if you find it to be incomplete, notify the proctor. Do all your supporting calculations in this booklet. In case of a doubtful mark on your answer card, your booklet can then be checked.. When you mark your card, use a soft lead pencil (#2). Erase fully any answers you want to change. Each of Problems 1 through 25 are worth 4 points apiece for a total of 100 points.

You may use a 5 x 8 note card and any scientific calculator other than a CAS calculuator. Thus, among Texas Instrument calculators, the TI-83, TI-83+, TI-85, and TI-86 are all fine but the TI-89 and TI-92 ARE NOT ALLOWED.

(1) Which of the following is an anti-derivative of xe^{-x}?

(A) $-xe^{-x}$
(B) $xe^{-x}+e^{-x}$
(C) $-xe^{-x}+e^{-x}$
(D) $-xe^{-x}-e^{-x}$
(E) $x^2e^{-x}/2$
(F) $-x^2e^{-x}/2+e^{-x}$
(G) $-x^2e^{-x}/2+xe^{-x}-e^{-x}$
(H) $x^2e^{-x}/2-xe^{-x}+e^{-x}$
(I) $x^2e^{-x}/2+xe^{-x}$
(J) $-x^2e^{-x}/2+xe^{-x}$

By integration by parts with

$u = x$, $dv = e^{-x}dx$, $v = -e^{-x}$

and

$$\int xe^{-x}dx = \int u\,dv = uv - \int v\,du$$
$$= -xe^{-x} + \int e^{-x}dx$$
$$= -xe^{-x} - e^{-x} + C$$

so $-xe^{-x} - e^{-x}$ is an anti-derivative of xe^{-x}.

(2) Which of the following integrals calculates the consumer's surplus (CS) for the demand curve $p = D(x) = 74 - x^2$ at the current selling price $\bar{p} = 10$?

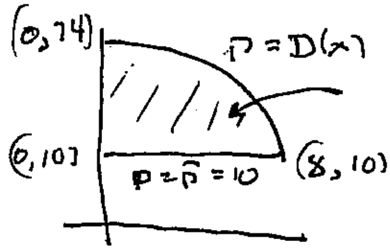
(A) $\int_0^{10}(74 - x^2)\,dx$
(B) $\int_0^{8}(64 - x^2)\,dx$
(C) $\int_0^{12}(82 - x^2)\,dx$
(D) $\int_0^{8}(74 - x^2)\,dx$
(E) $\int_0^{10}(64 - x^2)\,dx$
(F) $\int_0^{10}(x^2 - 64)\,dx$
(G) $\int_0^{10}(64 + x^2)\,dx$
(H) $\int_0^{12}(82 + x^2)\,dx$
(I) $\int_0^{6}(64 - x^2)\,dx$
(J) $\int_0^{6}(x^2 - 64)\,dx$

The demand $\bar{x}$ for which $D(\bar{x}) = \bar{p} = 10$ satisfies

$10 = 74 - \bar{x}^2$

or $\bar{x}^2 = 74 - 10 = 64$

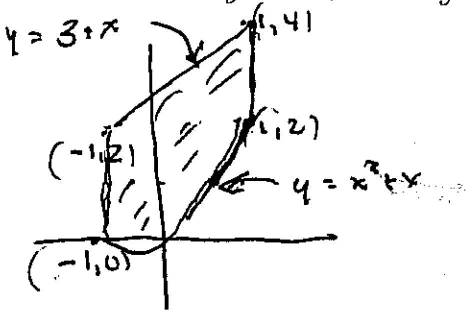
so $\bar{x} = 8$



$$CS = \int_0^8 (D(x) - 10)\,dx = \int_0^8 (64 - x^2)\,dx$$

(3) What is the area between the curves $y = x^2 + x$ and $y = 3 + x$ for $-1 \le x \le 1$?

(A) 7/3
(B) 8/3
(C) 3
(D) 10/3
(E) 11/3
(F) 4
(G) 13/3
(H) 14/3
(I) 5
(J) 16/3



Area between the curves

$$= \int_{-1}^{1} \{(3 + x) - (x^2 + x)\}\,dx$$

$$= \int_{-1}^{1} (3 - x^2)\,dx$$

$$= \left(3x - \frac{x^3}{3}\right)\Big|_{x=-1}^{x=1}$$

$$= 6 - 2/3$$

$$= \boxed{16/3}$$

(4) When a steady income stream of \$20,000 per year is deposited into an account on which interest is compounded continuously at the annual rate of 5% per year, what is the future value (FV) of the account 20 years from the time the account is opened?

(A) $\$20,000e$
(B) $\$200,000(1-e^{-1})$
(C) $\$200,000e(1+e^{-1})$
(D) $\$40,000e^2$
(E) $\$400,000(e^2-1)$
(F) $\$400,000(e-e{-}1)$
(G) $\$800,000e$
(H) $\$400,000(e-1)$
(I) $\$400,000(e-e^{-2})$
(J) $\$160,000e^2$

$$FV = \int_0^{20} 20{,}000\, e^{(20-t)(.05)}\,dt$$
$$= 20{,}000 \int_0^{20} e^{(1-t/20)}\,dt$$
$$= 20{,}000\, e \int_0^{20} e^{-t/20}\,dt$$
$$= 20{,}000\, e \left(\frac{-e^{-t/20}}{1/20}\right)\Big|_0^{t=20}$$
$$= \frac{20{,}000}{1/20}\, e\,(-e^{-1}+1)$$
$$= \cancel{400{,}000}\ 400{,}000\,(e-1)$$

(5) Evaluate $(\partial f/\partial x)(2,3) = f_x(2,3)$ for $f(x,y) = ye^{-2xy}$.

(A) $12e^{-6}$
(B) $-6e^{-6}$
(C) $-18e^{-12}$
(D) $6e^{-12}$
(E) $-9e^{-12}$
(F) e^{-12}
(G) $2e^{-12}$
(H) $12e^{-12}$
(I) $6e^{-9}$
(J) $-6e^{-9}$

For any (x,y)

$$f_x(x,y) = y\,\frac{\partial}{\partial x}\left(e^{-2xy}\right)$$
$$= y\left(-2y\,e^{-2xy}\right)$$
$$= -2y^2 e^{-2xy}$$

so

$$f_x(2,3) = -2(9)\,e^{-2(6)}$$
$$= -18\,e^{-12}$$

(6) Which of the following are the critical points for the function $f(x,y) = x^3 - 3xy^2 + 6y$?

(A) $(0,0)$ and $(-1,-1)$
(B) $(0,0)$ and $(1,1)$
(C) $(0,1)$ and $(-1,0)$
(D) $(1,1)$ and $(2,2)$
(E) $(2,2)$ and $(-2,-2)$
(F) $(1,1)$ and $(-1,-1)$
(G) $(1,2)$ and $(2,1)$
(H) $(0,0), (1,1)$, and $(-1,-1)$
(I) $(0,0), (1,1)$, and $(2,2)$
(J) $(1,1), (-1,-1), (2,2)$, and $(-2,-2)$

$0 = f_x = 3x^2 - 3y^2$ when $x^2 = y^2$

$0 = f_y = -6xy + 6$ when $xy = 1$

The solutions of the two equations $x^2 = y^2$, $xy = 1$ are $x = y = 1$ and $x = y = -1$

so $(1,1)$ and $(-1,-1)$ are the critical point of f

(7) The function $f(x,y) = x^3/3 + y^3/3 - 6xy + 15$ has $(0,0)$and $(6,6)$ as critical points. Which of the following describes the behaviour of f near these two points?

(A) Both are saddle points for f.
(B) Both are local minimum points for f.
(C) Both are local maximum points for f.
(D) These points are neither local max/min nor saddle points for f.
(E) $(0,0)$ is a local minimum point for f, $(6,6)$ is a saddle point.
(F) $(0,0)$ is a saddle point for f, $(6,6)$ is a local minimum point.
(G) $(0,0)$ is a local maximum point for f, $(6,6)$ is a saddle point.
(H) $(0,0)$ is a saddle point for f, $(6,6)$ is a local maximum point.
(I) $(0,0)$ is a local minimum point for f, $(6,6)$ is a local maximum point.
(J) $(0,0)$ is a local maximum point for f, $(6,6)$ is a local minimum point.

$f_x = x^2 - 6y$ and $f_y = y^2 - 6x$

so $f_{xx} = 2x$, $f_{xy} = f_{yx} = -6$, $f_{yy} = 2y$

Critical Point	$A = f_{xx}$	$B = f_{xy}$	$C = f_{yy}$	Sign $(AC - B^2)$	Type of critical point
$(0,0)$	0	-6	0	< 0	saddle point
$(6,6)$	$12 > 0$	-6	12	> 0	local minimum

(8) What is the maximum value of $f(x,y) = 4xy$ subject to the constraint $x^2 + y^2 = 4$

(A) 4
(B) $4\sqrt{3}$
(C) $6\sqrt{2}$
(D) $8\sqrt{3}$
(E) $8\sqrt{2}$
(F) $12\sqrt{2}$
(G) $6\sqrt{3}$
(H) $4\sqrt{5}$
$\boxed{\text{(I) } 8}$
(J) 16

With $g(x,y) = x^2+y^2$,
the Lagrange multiplier equations reduce to
$8y^2 = f_x g_y = f_y g_x = 8x^2$
and $x^2+y^2=4$
Then $x^2 = y^2 = 2$ so
$(x,y) = \pm(\sqrt{2},\sqrt{2})$ or $\pm(\sqrt{2},-\sqrt{2})$
The value of f at $\pm(\sqrt{2},\sqrt{2})$ is
$\cancel{4}\cdot(\sqrt{2})^2 = 8$
while the value of f at $\pm(\sqrt{2},-\sqrt{2})$ is
-8 so 8 is the maximum value of f
subject to the constraint $x^2+y^2=4$

(9) Use your calculator to find to two decimal places the slope of the line $y = ax + b$ which best fits the 6 data points (1,3), (2,4),(3,6), (4,7), (5,8), and (6,9) in the sense of minimizing the sum of the squares of the residual errors in the y-values.

(A) 1.11
(B) 1.14
(C) 1.17
(D) 1.20
$\boxed{\text{(E) } 1.23}$
(F) 1.26
(G) 1.29
(H) 1.32
(I) 1.35
(J) 1.38

Putting the x-data $\{1,2,3,4,5,6\}$ in L1
and the y-data $\{3,4,6,7,8,9\}$ in L2
and using
STAT CALC 4 LinReg(ax+b) L1, L2
gives $a = 1.228\ldots \approx 1.23$

(10) Evaluate $\int\int_R f(x,y)dA$ for $f(x,y) = ye^{xy}$ and $R = [0,2] \times [1,4]$.

(A) $e^4 - 2e^2 + 1$
(B) $e^8 - 2e^4$
(C) $e^8 - 3$
(D) $e^8/2 - e^2/2 - 3$
(E) $e^4/2 - 2$
(F) $e^8 - 2e^4 + 1$
(G) $e^8/2 - e^4/2 + 1$
(H) $e^8/2 - e - 3$
(I) $e^4 - 4e + 3$
(J) $e^8 - e^2 - 1$

$$\iint_R f(x,y)\,dA = \int_{y=1}^{4}\left(\int_{x=0}^{2} ye^{xy}dx\right)dy = \int_{y=1}^{4}\left(e^{xy}\Big|_{x=0}^{x=2}\right)dy = \int_{y=1}^{4}(e^{2y}-1)dy = \left(\frac{1}{2}e^{2y} - y\right)\Big|_1^4 = \boxed{\frac{1}{2}e^8 - \frac{1}{2}e^2 - 3}$$

(11) Find the general solution of the differential equation $y' = \frac{2x}{3y^2}$.

(A) $y = (x^2 + C)^{1/3}$
(B) $y = (x^3 + C)^{1/2}$
(C) $]y = (x^{2/3} + C)$
(D) $]y = 2(x^2 + C)/3y^3$
(E) $]y = (x^2 + C)/3y$
(F) $]y^2 = (x^3 + C)^{1/6}$
(G) $]y^3 = (x^2 + C)^{1/6}$
(H) $]y = (x^2 + C)^{2/3}$
(I) $]y = (x + C)^{2/3}$
(J) $]y = (x + C)^{1/6}$

$$\frac{dy}{dx} = y' = 2x/3y^2$$
$$3y^2dy = 2x\,dx$$
$$y^3 = \int 3y^2dy = \int 2x\,dx = x^2 + C$$
$$y = (x^2+C)^{1/3}$$

(12) The number $S(t)$ of millions of life insurance policies sold by a particular company in year t of company operations satisfies the limited growth equation $dS/dt = k(20 - S)$ where k is a proportionality constant. If $S(0) = 0$ and $S(10) = 5$, what is $S(20)$?

(A) 7.5
$\boxed{\text{(B) 8.75}}$
(C) 10
(D) 11.25
(E) 11.75
(F) 12.25
(G) 12.5
(H) 13
(I) 13.5
(J) 13.75

Since $\frac{d}{dt}(20-S) = -\frac{dS}{dt} = -k(20-S)$

$20 - S = (20 - S(0))e^{-kt} = 20e^{-kt}$

when $t = 10$, $S(10) = 5$ so

$15 = 20 - 5 = 20e^{-10k}$

with $e^{-10k} = 15/20 = 3/4$

Then for $t = 20$, $e^{-20k} = (3/4)^2$

and $20 - S(20) = (20)(3/4)^2$

or $S(20) = 20 - 20(3/4)^2$

$= \boxed{8.75}$

(13) Find the unique solution of the differential equation $y' + 2y/x = 1/x^2$ for which $y(1) = 2$.

$\boxed{\text{(A) } y = (x+1)/x^2}$
(B) $y = (x+1)^{1/2}$
(C) $y = (x^2+3)/(x+1)$
(D) $y = \ln x + 2 - 1/x$
(E) $y = .5\ln(x+1)$
(F) $y = (x+1)/x$
(G) $y = 2e^(x - 1)$
(H) $y = x2e^x$
(I) $y = -1/(x \ln x)$
(J) $y = x^2 \ln x$

We use the integrating factor

$I(x) = e^{\int \frac{2\,dx}{x}} = e^{2\ln x} = x^2$

multiplying the equation by x^2 converts it to

$\frac{d}{dx}(x^2 y) = x^2 y' + 2xy = 1$

So $x^2 y = x + C$

Putting $x = 1$ and $y = 2$

gives $2 = 1 + C$ or $C = 1$

Then $\boxed{y = \frac{x+1}{x^2}}$

(14) What is the coefficient of x^2 in the Taylor series at 0 for $f(x) = \frac{1}{(1-x)^2}$?

(A) -2
(B) -1
(C) 1
(D) 2
(E) 3
(F) 4
(G) 0
(H) 1/2
(I) -1/2
(J) -3

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + \cdots$

$\boxed{a_2 = 3}$

(15) What is the coefficient of x^3 in the Taylor series at 0 for the function $\ln(1 + 2x)$.

(A) -16/3
(B) -8/3
(C) -4/3
(D) -1/3
(E) -2
(F) 2
(G) 1/3
(H) 4/3
(I) 8/3
(J) 16/3

$\ln 1+t = t - t^2/2 + t^3/3 + \cdots$

$\ln(1+2x) = 2x - \frac{4x^2}{2} + \frac{8x^3}{3} + \cdots$

$\boxed{a_3 = 8/3}$

(16) What is the radius of convergence of the power series at 0 for the function $1/(1+4x)$

$\boxed{\text{(A) } 1/4}$
(B) 1/2
(C) 3/4
(D) 1
(E) 5/4
(F) 3/2
(G) 7/4
(H) 2
(I) 9/4
(J) 9/2

$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$

converges for $|t| < 1$

Putting $t = 4x$

$\frac{1}{1+4x} = 1 - 4x + (4x)^2 + \cdots$

for $|4x| < 1$

or $|x| < 1/4$

(17) For $f(x) = x^5/(1+x)^{15}$, what is the smallest value of n for which the coefficient of x^n in the Taylor series of f at 0 is not zero?

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
$\boxed{\text{(F) } 5}$
(G) 6
(H) 10
(I) 15
(J) 20

For $g(x) = \frac{1}{(1+x)^{15}}$

$g(0) = 1$ so g has the

Taylor series

$g(x) = 1 + \frac{g'(0)}{1} x + \frac{g''(0)}{2!} x^2 + \cdots$

Then $f(x) = x^5 g(x) = x^5 + \frac{g'(0)}{1} x^6 + \cdots$

with $1 =$ coefficient of x^5 is not 0

and ~~lower~~ smaller powers of x don't

appear.

(18) Use the fourth order Taylor polynomial at 0 for $f(x) = e^{-.1x^2}$ to approximate the integral $\int_0^1 e^{-.1x^2}\,dx$.

(A) $1 - 1/3 + 1/100$
(B) $1 + 1/30 - 1/1,000$
(C) $1 + 1/30 + 1/1,000$
(D) $1 - 1/30 + 1/1,000$
(E) $1 - 1/30 - 1/10000$
(F) $1 - 1/3 + 1/500$
(G) $1 + 1/30 + 1/500$
(H) $1 - 1/30 + 1/500$
(I) $1 - 1/2 + 1/5000$
(J) $1 - 1/300 + 1/50000$

$$e^{-t} = 1 - t + \frac{t^2}{2} + \cdots$$

$$e^{-x^2/10} = 1 - \frac{x^2}{10} + \frac{x^4}{2(100)} + \cdots$$

$$\int_0^1 e^{-x^2/10}\,dx \approx \int_0^1 \left(1 - \frac{x^2}{10} + \frac{x^4}{2(100)}\right)dx$$

$$= \left(x - \frac{x^3}{30} + \frac{x^5}{1000}\right)\Big|_0^1$$

$$= 1 - \frac{1}{30} + \frac{1}{1000}$$

(19) Let X be a random variable with probability density function given by $f(x) = 4/x^5$ for $x > 1$ and $f(x) = 0$ for $x \le 1$. For $x > 1$, which of the following is the cumulative distribution function $F(x)$ of X.

(A) $1 - 1/x^8$
(B) $1 - 1/x^6$
(C) $1/x^5$
(D) $1 - 1/x^5$
(E) $1 - 4/x^4$
(F) $20/x^6$
(G) $4/x^4$
(H) $4 - 4/x^4$
(I) $1 - 1/x^4$
(J) $1/x^4$

For $x > 1$

$$F(x) = \int_1^x f(t)\,dt$$

$$= \int_1^x \frac{4\,dt}{t^5}$$

$$= \frac{-4}{4t^4}\Big|_1^x$$

$$= 1 - \frac{1}{x^4}$$

(20) At a fast food restaurant, waiting times for orders to be filled are exponenentially distributed with a mean waiting time of 4 minutes. Which of the following expressions is the probability that an order will be filled between 3 and 5 minutes?

(A) $4\ln 2$
(B) $e^{12} - e^{15}$
(C) $e^{-12} - e^{-15}$
(D) $e^{-4/5} - e^{-4/3}$
(E) $e^{-4/3} - e^{-4/5}$
(F) $e^{-3/(4\ln 2)} - e^{-5/(4\ln 2)}$
(G) $e^{-3/4} - e^{-5/4}$
(H) $e^{-3/4} + e^{-5/4} - 2$
(I) $e^{5/4} - e^{3/4}$
(J) $e^{-3/8} - e^{-5/8}$

X = waiting time random variable has p.d.f.

$$f(x) = \begin{cases} \frac{1}{4} e^{-x/4} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and c.d.f. $F(x) = 1 - e^{-x/4}$ for $x > 0$

$$P(3 \leq X \leq 5) = F(5) - F(3) = -e^{-5/4} + e^{-3/4}$$

(21) Waiting times for tax refund cehcks are exponentially distributed with an average (mean) waiting time of 30 days. Rounded off to the nearest whole day, what is the median waiting time?

(A) 15
(B) 16
(C) 17
(D) 18
(E) 19
(F) 20
(G) 21
(H) 22
(I) 23
(J) 24

λ = mean = 30

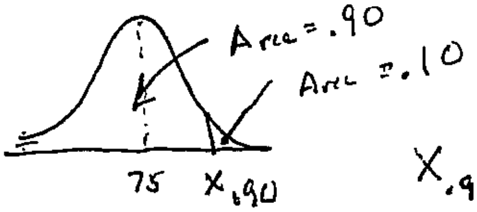
m = median = $\lambda \ln 2$

$= (30)(.693\ldots)$

$\approx$ 21 days

(22) Suppose scores on an exam are normally distributed with mean 75 and standard deviation 10. To the nearest tenth, what is the 90th percentile score, i.e., the score dividing the lower 90% from the upper 10%.

(A) 82.6
(B) 83.1
(C) 84.0
(D) 84.3
(E) 84.9
(F) 85.4
(G) 86.7
$\boxed{\text{(H) 87.8}}$
(I) 88.5
(J) 90.0

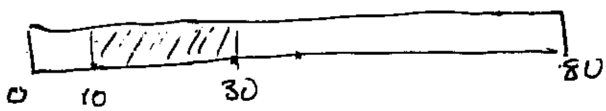


By the TI-83

$$X_{.90} = \text{invnorm}(.9, 75, 10) = 87.81\ldots \approx 87.8$$

(23) Suppose the positions of sweet gum balls dropping on a narrow driveway of length 80 feet are uniformly distributed What is the probability that a gum ball will drop between 10 and 30 feet from the back end of the driveway?

(A) .20
(B) .22
$\boxed{\text{(C) .25}}$
(D) .28
(E) .33
(F) .50
(G) .66
(H) .75
(I) .80
(J) 1.00



X = position random variable has p.d.f $f(x) = \begin{cases} 1/80 & \text{for } 0 \le x \le 80 \\ 0 & \text{otherwise} \end{cases}$

$$P(10 \le X \le 30) = \int_{10}^{30} f(x)\,dx = \frac{20}{80} = .25$$

(24) Machine bolts of a certain type are desired to have a length of 5 cm. but are considered acceptable if their lengths are between 4.95 cm. and 5.05 cm. Suppose the manufacturing process produces bolts whose lengths are normally distributed with a mean of 5 cm. and a standard deviation of .03 cm. To the nearest tenth of one percent, what percentage of the bolts produced are acceptable?

(A) 87.6%
(B) 88.2%
(C) 88.9%
(D) 89.7%
(E) 90.4%
(F) 91.1%
(G) 92.3%
(H) 93.0%
(I) 93.8%
(J) 94.5%

The probability of a bolt having a length between 4.95 and 5.05 is given by

normcdf = .9044...

This means 90.4% of the bolts are acceptable

(25) Recovery times for a certain flu strain are exponentially distributed with a median recovery time of 3 days. What percentage of those afflicted with this flu strain take more than 6 days to recover?

(A) 13.4%
(B) 14.3%
(C) 15.2%
(D) 16.5%
(E) 18.1%
(F) 19.8%
(G) 20.6%
(H) 21.7%
(I) 22.9%
(J) 25.0%

$m = \lambda \ln 2 = 3$

so the recovery time random variable X has c.d.f.

$$F(x) = 1 - e^{-x/\lambda} = 1 - e^{-x/3/\ln 2}$$

Then $P(X > 6) = 1 - F(6)$

$$= e^{-\frac{6\ln 2}{3}}$$

$$= e^{\frac{1}{2\ln 2}} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

So 25% of those afflicted take > 6 days to recover