Math 128, Spring 2005
Section A, Quiz 9

Name: Solutions

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Please write down all relevant mathematics. You have 20 minutes.

1. PROBLEM ONE

What are the interval and radius of convergence of the Taylor series at $a = 3$ for the function $f(x) = 5(11-2x)^{-3}$. Hint: $11 - 2x = 5 - 2(x-3) = 5(1 - \frac{2(x-3)}{5})$.

Taylor Series for .

$$(1+t)^{\alpha} = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2}t^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!}t^n + \cdots$$

with $|t|<1$ as interval of convergence

so setting $t = \frac{2(x-3)}{5}$,

$$|t| < 1$$

$$\left|\frac{2(x-3)}{5}\right| < 1$$

$|x-3| < \frac{5}{2}$ → Radius

$-\frac{5}{2} < x-3 < \frac{5}{2} \Rightarrow \boxed{\frac{1}{2} < x < \frac{11}{2}}$ ← interval of convergence.

2. PROBLEM TWO

Use the second order Taylor polynomial at 0 for the function $f(x) = e^{-x^2/40}$ to approximate $\int_0^1 f(x)\,dx$.

For e^t, 1st degree is $1+t$

for $e^{-x^2/40}$, 2nd degree is

$$p_2(x) = 1 - \frac{x^2}{40}$$

$$\int_0^1 p_2(x)\,dx = \int_0^1 1 - \frac{x^2}{40}\,dx$$

$$= x - \frac{x^3}{120} = 1 - \frac{1}{120}$$

3. PROBLEM THREE

For $f(x)$ as in Problem 2, estimate the error in your approximation using the first term omitted in term-by-term integration of the Taylor series for f.

1st omitted term is

$$\underbrace{\boxed{1 - \frac{x^2}{40}}}_{p_2(x)} + \underbrace{\frac{x^4}{2 \cdot 1600}}_{\text{1st omitted term}} + \cdots$$

Error $|f(x) - p_2(x)| \le \frac{x^4}{2 \cdot 1600}$

$$\int_0^1 |f(x) - p_2(x)|\,dx \le \int_0^1 \frac{x^4}{2 \cdot 1600}\,dx$$

$$= \frac{x^5}{2 \cdot 5 \cdot 1600}\Big|_0^1 = \frac{1}{2 \cdot 8000} = \frac{1}{16{,}000}$$

Math 128, Spring 2005
Section B, Quiz 9 Name: ____________________

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Please write down all relevant mathematics. You have 20 minutes.

1. PROBLEM ONE

What are the interval and radius of convergence of the Taylor series at $a = 2$ for the function $f(x) = 8(9-2x)^{-4}$. Hint: $9 - 2x = 5 - 2(x-2) = 5(1 - \frac{2(x-2)}{5})$.

Making substitution $t = \frac{2(x-2)}{5}$ for Taylor series $\frac{1}{1-t}$, which has interval of convergence $|t| < 1$,

$\left|\frac{2(x-2)}{5}\right| < 1$

$|x-2| < \frac{5}{2} \longrightarrow$ Radius $= 5/2$

$-5/2 < x-2 < 5/2$

$-\frac{1}{2} < x < 9/2$. Interval of Convergence.

2. PROBLEM TWO

Use the second order Taylor polynomial at 0 for the function $f(x) = e^{-x^2/10}$ to approximate $\int_0^1 f(x)\,dx$.

2nd order: use 1st order T.P. for e^t which is $1+t$.

So $P_2(x) = 1 - \frac{x^2}{10}$,

$\int_0^1 f(x)\,dx \approx \int_0^1 P_2(x)\,dx$

$= \int_0^1 1 - \frac{x^2}{10}\,dx = x - \frac{x^3}{30}\Big|_0^1$

$= 1 - \frac{1}{30}$.

3. PROBLEM THREE

For $f(x)$ as in Problem 2, estimate the error in your approximation using the first term omitted in term-by-term integration of the Taylor series for f.

First term omitted: is term corresponding to $\underset{t = -\frac{x^2}{10}}{\frac{t^2}{2}} = \frac{x^4}{200}$

So, Error $\cancel{}$

$|f(x) - P_2(x)| < \frac{x^4}{200}$

So, $\int_0^1 |f(x) - P_2(x)|\,dx < \int_0^1 \frac{x^4}{200}$

$= \frac{x^5}{1000}\Big|_0^1 = \frac{1}{1000}$

Math 128, Spring 2005
Section C, Quiz 9

Name: ____________________

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Please write down all relevant mathematics. You have 20 minutes.

1. PROBLEM ONE

What are the interval and radius of convergence of the Taylor series at $a = 2$ for the function $f(x) = 12(13 - 5x)^{-5}$. Hint: $13 - 5x = 3 - 5(x-2) = 3(1 - \frac{5(x-2)}{3})$.

Substituting $t = \frac{5(x-2)}{3}$, using Taylor Series for $\frac{1}{1-t}$, good for $|t| < 1$,

$$\left|\frac{5(x-2)}{3}\right| < 1$$

$|x-2| < \frac{3}{5}$, Radius $= \frac{3}{5}$

$-\frac{3}{5} < x-2 < \frac{3}{5}$

$\frac{7}{5} < x < \frac{13}{5}$ → interval of Convergence.

2. PROBLEM TWO

Use the second order Taylor polynomial at 0 for the function $f(x) = ln(16 + x^2)$ to approximate $\int_0^1 f(x)\,dx$.

$\ln(16+x^2) = \ln(16(1+(\frac{x}{4})^2))$

$= \ln 16 + \ln(1+(\frac{x}{4})^2)$

1st order Taylor polynomial for $\ln(1+t)$ is t

Substitute $t = (\frac{x}{4})^2$

So $p_2(x) = \ln 16 + (\frac{x}{4})^2$

$\int_0^1 f(x)\,dx \sim \int_0^1 p_2(x)\,dx$

$= \int_0^1 \ln 16 + \frac{x^2}{16}\,dx$

$= x\ln 16 + \frac{x^3}{48}\Big|_0^1 = \ln 16 + \frac{1}{48}$

3. PROBLEM THREE

For $f(x)$ as in Problem 2, estimate the error in your approximation using the first term omitted in term-by-term integration of the Taylor series for f.

Omitted term: corresponds to term $-\frac{t^2}{2}$ in Taylor Series for $\ln(1+t)$

So, since we substituted $t = (\frac{x}{4})^2$, Omitted term is

$-\left(\left(\frac{x}{4}\right)^2\right)^2 = -\frac{x^4}{256}$

Error is $|f(x) - p_2(x)| \le \frac{x^4}{2\cdot 256}$

So, Error in Estimate is

$\int_0^1 |f(x) - p_2(x)|\,dx < \int_0^1 \frac{x^4}{2\cdot 256}\,dx$

$= \frac{x^5}{2\cdot 1280}\Big|_0^1 = \frac{1}{(1280)(2)}$