Name:
ID:
Discussion Section:

This exam has 15 questions:

- 12 multiple choice worth 5 points each.
- 3 hand graded worth 40 points total.

Important:

- No graphing calculators!
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. You will be graded on the ease of reading your solution.
- You are allowed a 3 × 5 note card for the exam.

1. Solve the initial value problem $y' = y + 2t$, $y(0) = 0$. $y' - y = 2t.$

   $I(t) = \exp\left(\int -1\,dt\right) = e^{-t}.$

   $\frac{d}{dt}\left(e^{-t}y\right) = 2te^{-t}$

   $e^{-t}y = 2\int te^{-t}\,dt =$

   $u = t \quad dv = e^{-t}dt$

   $du = dt \quad v = -e^{-t}$

   $e^{-t}y = -2te^{-t} - 2e^{-t} + C.$

   $y = -2t - 2 + Ce^{t}$

   $0 = y(0) = -2 + C.$

   $C = 2.$

   (a) $y = -2t$
   (b) $y = t^2$
   (c) $y = \frac{3t^2}{2}$
   (d) $y = e^{t^2} - 1$
   (e) $y = -2 - 2t + 2e^t$
   (f) $y = -2 + 2t + 2e^{-t}$
   (g) $y = e^t + t^2 - 1$
   (h) $y = (-2t - 2)e^{-t} + 2$

2. Find an integrating factor $I(t)$ for the equation $ty' + y = t^2$. $y' + \frac{1}{t}y = t.$

   $I(t) = \exp\left(\int \frac{1}{t}\,dt\right) = e^{\ln t} = t.$

   (a) $I(t) = t$
   (b) $I(t) = t^2$
   (c) $I(t) = \frac{t^2}{2}$
   (d) $I(t) = \frac{t^3}{3}$
   (e) $I(t) = e^t$
   (f) $I(t) = e^{-t}$
   (g) $I(t) = e^{t^2}$
   (h) $I(t) = \ln t$

3. Suppose that you open a bank account that earns 5% interest compounded continuously and you initially deposit \$1000 into the account. After the initial deposit, you continuously contribute to the account at a rate of $500+100t$ dollars per year. Which of the following initial value problems would you solve in order to find the amount $y(t)$ in the account after t years?

   (a) $y' = .05y(500+100t)$, $y(0) = 1000$
   (b) $y' = .05y + 500 + 100t$, $y(0) = 1000$
   (c) $y' = .05y + 1000$, $y(0) = 500$
   (d) $y' = .05y + 1500 + 100t$, $y(0) = 0$
   (e) $y' = .05y + 1500 + 100t$, $y(0) = 500$
   (f) $y' = e^{.05t} + 500 + 100t$, $y(0) = 1000$
   (g) $y' = (500+100t)e^{.05t}$, $y(0) = 1000$
   (h) $y' = 1000e^{.05t} + 500 + 100t$, $y(0) = 1500$

$y(0) = 1000$

$y' = \underbrace{.05y}_{\text{interest}} + \underbrace{500 + 100t}_{\text{deposits}}.$

4. A tank contains 1000L of water. A solution containing .01kg/L salt enters the tank at a rate of 20L/minute. The solution mixes with the water in the tank, and the mixture exits the tank at the same rate of 20L/minute. Which of the following differential equations is satisfied by the amount (in kg) of salt in the tank at time t?

   (a) $y' = .01 - 20y$
   (b) $y' = .01t - 20y$
   (c) $y' = .01y + 1000 - 20t$
   (d) $y' = .2 - .02y$
   (e) $y' = .2 - 20y$
   (f) $y' = 10 - 20000y$
   (g) $y' = -19.99$
   (h) $y' = e^{.01t} - 20$

$A =$ amt of salt @ time t.

Rate in: $\frac{.01\,\text{kg}}{L} \times \frac{20\,L}{\text{min}} = .2\,\text{kg}/\text{min}.$

Rate out: $\underbrace{\left(\frac{A}{1000\,L}\right)}_{\text{concentration}} \frac{20\,L}{\text{min}} = \frac{20A}{1000} = \frac{A}{50}\,\text{kg}/\text{min}$

$A' = .2 - \frac{A}{50}.$

5. Suppose that $y(t)$ satisfies the initial value problem $y' = 2t + y^2$, $y(0) = 1$. Use Euler's method with step size .5 to estimate $y(1)$.

   (a) 0.5
   (b) 1
   (c) 1.5
   (d) 2
   (e) 2.5
   (f) 2.625
   (g) 3.125
   (h) 7

t	y	y'	Approximating line
0	1	1	$y = 1 + t.$
.5	$\frac{3}{2}$	$1+\frac{9}{4} = \frac{13}{4}$	$y = \frac{3}{2} + \frac{13}{4}(t - \frac{1}{2}).$

$y(1) \approx \frac{3}{2} + \frac{13}{4}\left(\frac{1}{2}\right) = 3.125$

6. Find the third Taylor polynomial of $\ln x$ at $x = 1$.

   (a) $x - \frac{1}{2}x^2 + \frac{1}{3}x^3$
   (b) $x - x^2 + 2x^3$
   (c) $1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3$
   (d) $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$
   (e) $(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3}(x-1)^3$
   (f) $(x-1) - (x-1)^2 + (x-1)^3$
   (g) $(x-1) - (x-1)^2 + 2(x-1)^3$
   (h) $\frac{1}{x}(x-1) - \frac{1}{2x^2}(x-1)^2 + \frac{1}{3x^3}(x-1)^3$

$f(x) = \ln(x)$ $\quad f(1) = 0$

$f'(x) = \frac{1}{x}$ $\quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\quad f''(1) = -1$

$f^{(3)}(x) = \frac{2}{x^3}$. $\quad f^{(3)}(1) = 2$

$P_3(x) = (x-1) + \frac{-1}{2}(x-1)^2 + \underbrace{\frac{2}{3!}}_{\frac{1}{3}.}(x-1)^3.$

7. Find the sum of the geometric series $4 + 3 + 2.25 + 1.6875 + \ldots$.

   (a) $\frac{1}{4}$
   (b) 4
   (c) $\frac{16}{3}$
   (d) 11
   (e) 12
   (f) 15
   (g) 16
   (h) 24
   (i) e^4
   (j) The series does not converge.

$= 4\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right)$

$= \frac{4}{1-\frac{3}{4}} = \frac{4}{\frac{1}{4}} =$ [redacted] 16

8. Consider a perpetuity that pays \$100 every year, with the first payment being made immediately. If the interest rate is 5% compounded annually, how much is the perpetuity worth?

   (a) \$95
   (b) \$105
   (c) \$952
   (d) \$1050
   (e) \$1900
   (f) \$2000
   (g) \$2100
   (h) The value of the perpetuity is infinite.

100 given in n years has present value:

$\frac{100}{(1.05)^n}$.

Value $= 100 + \frac{100}{(1.05)} + \frac{100}{(1.05)^2} + \cdots$

$= 100\left(1 + \frac{1}{1.05} + \frac{1}{(1.05)^2} + \cdots\right)$

$= 100\left(\frac{1}{1-\frac{1}{1.05}}\right) = 2100$

9. Which of the following series converge?

   I. $\sum_{n=1}^{\infty} \frac{1}{n}$ $\to$ diverges

   II. $\sum_{n=1}^{\infty} \frac{1}{2^n}$ geometric

   III. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ p-series $p = 3/2$.

   (a) none of them
   (b) only I
   (c) only II
   (d) only III
   (e) only I and II
   (f) only I and III
   (g) only II and III
   (h) all of them

10. Which of the following series converge?

   I. $\sum_{n=1}^{\infty} \frac{5^n}{4^n}$ $\to$ diverges,

   II. $\sum_{n=1}^{\infty} \frac{1}{n^3+3}$ $\to$ converges by comparison test to $\sum_{n=1}^{\infty} \frac{1}{n^3}$

   III. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$ $\to$ diverges p series with $p = \frac{1}{2}$

   (a) none of them
   (b) only I
   (c) only II
   (d) only III
   (e) only I and II
   (f) only I and III
   (g) only II and III
   (h) all of them

11. Which of the following statements describes an accurate use of the comparison test to determine whether $\sum_{n=1}^{\infty} \frac{1}{n4^n}$ converges?

(a) $\frac{1}{n4^n} < \frac{1}{n}$ for all n, so $\sum_{n=1}^{\infty} \frac{1}{n4^n}$ converges.
(b) $\frac{1}{n4^n} < \frac{1}{n}$ for all n, so $\sum_{n=1}^{\infty} \frac{1}{n4^n}$ diverges.
(c) $\frac{1}{n4^n} > \frac{1}{n}$ for all n, so $\sum_{n=1}^{\infty} \frac{1}{n4^n}$ converges.
(d) $\frac{1}{n4^n} > \frac{1}{n}$ for all n, so $\sum_{n=1}^{\infty} \frac{1}{n4^n}$ diverges.
(e) $\frac{1}{n4^n} < \frac{1}{4^n}$ for all n, so $\sum_{n=1}^{\infty} \frac{1}{n4^n}$ converges.
(f) $\frac{1}{n4^n} < \frac{1}{4^n}$ for all n, so $\sum_{n=1}^{\infty} \frac{1}{n4^n}$ diverges.
(g) $\frac{1}{n4^n} > \frac{1}{4^n}$ for all n, so $\sum_{n=1}^{\infty} \frac{1}{n4^n}$ converges.
(h) $\frac{1}{n4^n} > \frac{1}{4^n}$ for all n, so $\sum_{n=1}^{\infty} \frac{1}{n4^n}$ diverges.

$\frac{1}{n4^n} < \frac{1}{4^n}$

$\sum_{n=1}^{\infty} \frac{1}{4^n}$ Converges

$\Rightarrow$ Series converges

12. Use the Taylor series expansion of $f(x) = \frac{x}{1-x^2}$ to determine what you would get if you took the 47th derivative of $f(x)$ and then evaluated it at $x = 0$.

(a) $f^{(47)}(0) = \frac{-47}{2208}$
(b) $f^{(47)}(0) = 0$
(c) $f^{(47)}(0) = \frac{1}{47!}$
(d) $f^{(47)}(0) = 1$
(e) $f^{(47)}(0) = 47$
(f) $f^{(47)}(0) = 2^{47}$
(g) $f^{(47)}(0) = 47!$
(h) $f^{(47)}(0) = 47^{47}$

$$\frac{x}{1-x^2} = x\left(\frac{1}{1-x^2}\right) = x\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} x^{2n+1}$$

$$\frac{f^{(47)}(0)}{47!} = 1$$ Coefficient of x^{47}.

Name:
ID:
Discussion Section:
WRITTEN PROBLEM—SHOW YOUR WORK

13. (a) Suppose that you put a thermometer in a fresh cup of coffee and find that it's at 90 degrees Celsius. After waiting one minute, you check the temperature again and find it has cooled to 80 degrees. If the temperature of the room is 20 degrees, determine how much longer you will have to wait before the coffee cools to a more drinkable 50 degrees.

14. (a) Find the Taylor series at $x = 0$ for $f(x) = e^{x^2}$.

    (b) Find a series that converges to the definite integral $\int_0^1 e^{x^2}\,dx$.

    (c) Sum the first four terms of the series to obtain an approximation of $\int_0^1 e^{x^2}\,dx$.

15. (a) Compute the third partial sum S_3 of the series $S = \sum_{n=1}^{\infty} \frac{1}{n^3}$.

    (b) Let R_3 be the error in using S_3 as an approximation of the total sum (In other words, $R_3 := S - S_3$.). Find an estimate that is *greater than* the true value of R_3 Then find another estimate that is *less than* the true value of R_3.

    (c) Find two numbers A and B such that $A \leq S \leq B$.

13 y = temp @ time t of coffee.

$y(0) = 90$.

$y(1) = 80$.

$T_0 = 20 =$ temp of surroundings.

find t s.t. $y(t) = 50$.

$y' = k(y-20)$. (k negative)

$$\int \frac{dy}{y-20} = \int k\,dt$$

$$\ln|y-20| = kt + C.$$

$y - 20 = Ae^{kt}$ ($A \in \mathbb{R}$eal Number).

$$y = 20 + Ae^{kt}.$$

$90 = y(0) = 20 + A \Rightarrow A = 70$.

$$y = 20 + \cancel{\boxed{Ae^{kt}}}\ 70e^{kt}$$

$$80 = y(1) = 20 + 70e^{k}$$

$$\frac{6}{7} = e^{k} \qquad k = \ln\left(\frac{6}{7}\right).$$

$$50 = y(t) = 20 + 70e^{\ln(\frac{6}{7})t}$$

$$\frac{30}{70} = \cancel{\boxed{e^{t}\left(\ln\left(\frac{6}{7}\right)\right)}}\ e^{t\ln\frac{6}{7}}$$

$$\ln\left(\frac{3}{7}\right) = t\ln\left(\frac{6}{7}\right)$$

$$t = \frac{\ln(3) - \ln 7}{\ln(6) - \ln(7)}$$

14 a) Taylor Series for e^t @ 0 is $\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

So for e^{x^2} it is:

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

b) $$\int_0^1 e^{x^2} dx = \int_0^1 \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} dx$$

$$= \sum_{k=0}^{\infty} \underbrace{\int_0^1 \frac{x^{2k}}{k!} dx}.$$

$$\frac{x^{2k+1}}{k!\,(2k+1)} \Big|_0^1 = \frac{1}{k!\,(2k+1)}$$

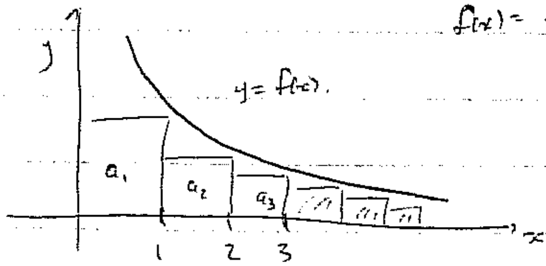
$$= \sum_{k=0}^{\infty} \frac{1}{k!\,(2k+1)}$$

c) $$1 + \frac{1}{1 \cdot 3} + \frac{1}{2(5)} + \frac{1}{3!\,(7)}$$

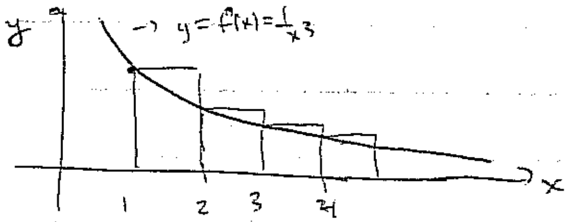
$$= 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} \approx 1.457$$

15 (a) $S_3 = \sum_{n=1}^{3} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} \simeq 1.16203$

(b) $f(x) = \frac{1}{x^3}$.



$R_3 \leq \int_3^\infty \frac{1}{x^3} dx. = \lim_{t \to \infty} -\frac{1}{2} \left[\frac{1}{x^2}\right]_3^t = \frac{1}{18}.$



$R_3 \geq \int_4^\infty \frac{1}{x^3} dx. = \lim_{t \to \infty} -\frac{1}{2} \left[\frac{1}{x^2}\right]_4^t = \frac{1}{32}.$

(c) $S - S_3 \doteq R_3$ satisfies

$\frac{1}{32} \leq S - S_3 \leq \frac{1}{18}.$

$S_3 + \frac{1}{32} \leq S \leq S_3 + \frac{1}{18}.$

$1.19328 \leq S \leq 1.21759..$