1. Recall that the graph of the equation $x + y = 1$ is the set of all points (x, y) that satisfy the equation. Graph the equation and find another equation that has NO common solution to the above equation. Describe geometrically what it means for these two linear equations to have no common solution.

2. Describe geometrically what it means for 2 linear equations in 2 variables to have exactly 1 solution. Find an example and graph the equations.

3. Consider the matrix equation

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   Describe in terms of solutions to linear equations what a solution vector to this matrix equation means. Draw geometrically what this means.

4. Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

   And again consider the matrix equation

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   Row reduce this system of linear equations to Echelon Form then describe geometrially what it means to row reduce the augmented matrix. Graph the resulting Row reduced matrix equations. Check that the solutions that you obtain before and after row reducing to Echelon form agree.

5. Consider the matrix that describes (counter-clockwise) rotation by an angle θ in $\mathbb{R}^2$:

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

   Suppose that you have 2 angles θ and φ. Describe (without computing) geometrically what the matrix product

$$R_\theta R_\varphi$$

   means. (Hint: You MUST use the notation in this problem).

6. Compute the matrix product

$$R_\theta R_\varphi$$

   and simplify (hint: it may be helpful to know some trigonometric identities). Why does this agree with your answer to the previous question?

7. Again, consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

   Describe geometrically what this matrix is as a composition of linear transformations. (Hint: think of the previous few problems and do a few easy examples). Explicitly write out the matrix A as a product of 2 matrices based on this answer.

8. Let
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad , \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
and consider the matrix equation
$$A\mathbf{x} = \mathbf{b}$$
(this is the same system of equations you studied in problems 3 and 4). Using your answer to problem 7, describe geometrically what the solution vector $\mathbf{x}$ represents.

9. Using your answer to the previous problem and using as few computations as possible, solve for the vector $\mathbf{x}$ geometrically. Your work must involve a series of pictures.

10. Based on your answers to the last 2 problems and without using the method in section 2.2 of the book, compute the inverse of the matrix A.

11. Determine the Column and Null space of the matrix A. Justify your answer by viewing the matrix as a linear transformation.

12. Are the columns of A linear independent? Justify your answer using at least 2 methods.

13. Suppose that we have a vector $\mathbf{x}$ and a number λ such that
$$A\mathbf{x} = \lambda\mathbf{x}.$$
Describe in your own words what this means geometrically and draw a picture exhibiting what this means. (The picture does not have to reflect what actually happens for this particular matrix A, just a picture that gives the idea of what the equation means).

14. Based on your answer to problem 7, why is it not feasible to find a vector $\mathbf{x} \in \mathbb{R}^2$ such that
$$A\mathbf{x} = \lambda\mathbf{x}$$
for some real number λ.

15. Compute the eigenvalue(s) and eigenvector(s) for $\mathbf{x}$. If you are unfamiliar with complex numbers please come and see me for assistance.

16. Based on your answer to problem 15, explain why this sheds more light on your answer to problem 14.