Research Statement

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1 Overview

The application of ideas in harmonic analysis to understand and solve problems in other areas of mathematics has led to much progress and proven fruitful. My research has primarily focused on adapting ideas in harmonic analysis to study function spaces and certain classes of operators. One part of my research has focused on developing the relationship between a class of operators on spaces of holomorphic functions and Fourier multipliers. In fact, questions about boundedness of this family of operators has an equivalent formulation in terms of boundedness of a class of multi-linear Fourier multipliers. The second part of my research is concerned with estimates on the size of the oscillation for boundary values of functions that give rise to Carleson measures for certain spaces of holomorphic functions. In the classical case of the Hardy spaces on the disc, Carleson measures are generated by $BMO(\mathbb{T})$ functions. Functions in $BMO(\mathbb{T})$ admit a restriction on their oscillation and the John-Nirenberg theorem gives a characterization of these functions in terms of that restriction. My research is concerned with the analogue of this result for the Dirichlet spaces (which are also sometimes referred to as analytic Besov spaces).

2 Multilinear Operators and Hankel Forms

It is a theorem of Lacey and Thiele [7] that the bilinear Hilbert transform admits L^p bounds. A special case of this theorem is the following: let $f = \sum_{n\geq 0} f_n z^n$, $g = \sum_{m\geq 0} g_n z^n \in \mathcal{H}^2(\mathbb{D})$ then

$$f \cdot g = \sum_{n,m\geq 0} f_n g_m z^{n+m} \in \mathcal{H}^1(\mathbb{D})$$

by Cauchy-Schwarz. Lacey and Thiele's theorem implies that

$$B(f, g) = \sum_{n\geq m\geq 0} f_n g_m z^{n+m} \in \mathcal{H}^1(\mathbb{D}).$$

This result is closely related to the triangular truncation of Hankel operators.

It is a classical result that the Hankel operator with symbol b is bounded if and only if $b \in BMO(\mathbb{T})$. We can think of a Hankel operator with symbol b in two equivalent ways, as a multiplier on the coefficients

$$H_b(f) \sim \sum_{n,m\geq 0} f_n b_{n+m} z^m$$

or as the infinite matrix given by

$$H_b = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ b_1 & b_2 & b_3 & b_4 & \cdots \\ b_2 & b_3 & b_4 & b_5 & \cdots \\ b_3 & b_4 & b_5 & b_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Define the triangular truncation of H_b as

$$\Pi(H_b) = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & b_2 & b_3 & b_4 & \cdots \\ 0 & 0 & b_4 & b_5 & \cdots \\ 0 & 0 & 0 & \ddots & \cdots \end{bmatrix}.$$

It turns out that boundedness of this matrix is equivalent to the boundedness of the operator

$$f \to \sum_{n \geq m} f_n b_{n+m} z^m.$$

In [3] and [6] it was independently proved that $\Pi(H_b)$ is bounded if and only if H_b is bounded if and only if $b \in BMO(\mathbb{T})$. The key to the non-trivial part of the proof is the reduction of the boundedness of $\Pi(H_b)$ to the boundedness of the bilinear Hilbert transform. My research has been concerned with obtaining an analogous result for the family of matrices

$$H_b^{\alpha,\beta} = \left[\frac{n^\alpha m^\beta}{(n+m)^{\alpha+\beta}} b_{n+m} \right]_{n,m \geq 0}$$

for $\alpha, \beta \geq 0$. These matrices are related to both the Calderon Commutator and Hankel operators on Bergman spaces. As one would expect, the key to this proof is the reduction to the boundedness of a bilinear singular Fourier multiplier on Hardy spaces. These multipliers have too large a singularity set for existing theorems about singular multi-linear multipliers on Lebesgue spaces to be applied directly. However, by taking advantage of the fact that Hardy space functions have vanishing negative Fourier coefficients, we are able to mollify the multiplier away from the Fourier supports to obtain an operator we can work with. A special case of my results is that the matrix $H_b^{1,0}$ is bounded if and only if $b \in BMO$. Another case is that $H_b^{\alpha,\beta}$ is bounded if and only if b is in the Bloch space. My results also extend to the Bergman spaces and I also give results for $p \neq 2$.

When one restricts attention to trace class Hankel operators on the Hardy space it is not known what condition on the symbol function characterize the boundedness of the truncation. Trace class Hankel operators are characterized by symbols belonging to the Besov space:

$$B_1 = \left\{ f \quad \text{analytic in} \quad \mathbb{D} : \int_{\mathbb{D}} |b''(z)| \, dA(z) < \infty \right\}$$

What is known is that if $\Pi(H_b)$ is trace class then $b \in B_1$ is necessary but not sufficient. In fact, any condition one imposes on the decay of the symbol b as you approach the boundary of $\mathbb{D}$ is too strong [3].

3 Carleson Measures, Oscillation and Besov Spaces

For a function f on $\mathbb{T}$ and interval I define:

$$m_I(f) = \frac{1}{|I|}\int_I f.$$

The John-Nirenberg theorem states that a function f is in $BMO(\mathbb{T})$ if and only if for all intervals I:

$$|\{x \in I : |f(x) - m_I(f)| > \lambda\}| \leq A|I| \exp\left(-\frac{B\lambda}{||f||_*}\right)$$

where $||f||_* := \sup_I \frac{1}{|I|}\int_I |f - m_I(f)|$ and A and B are absolute constants. These functions generate Carleson measures by: $f \in BMO(\mathbb{T})$ if and only if $|f'|^2(1-|z|^2)dxdy$ is a Carleson measure. The other part of my research is concerned with the study of the Dirichlet spaces on the disc. The Dirichlet spaces are defined as

$$B_a^p = \left\{ f \text{ holomorphic in } \mathbb{D} : \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^a dxdy < \infty \right\}.$$

The measure theoretic characterization of the Carleson measures for the Dirichlet spaces is in [1], where it is also shown that f is bounded and $|f'|^p dxdy(1-|z|^2)^{(p-2+a)}$ is a Carleson measure for B_a^p if and only if f is a multiplier for B_a^p. My measure theoretic research in this area is concerned with characterizing functions that give rise to Carleson measures for the Dirichlet space by proving an analogue of the John-Nirenberg Theorem. The proof (in the dyadic model) is by a stopping time argument just as the classic proof by John and Nirenberg. However, the more complicated Carleson condition in this context leads to a more intricate stopping time argument. The approach I have taken is to first prove the analogue of the John-Nirenberg theorem for a dyadic model of the Dirichlet spaces. The extension from the model case to the general case is a work in progress and I am optimistic. I have the following analogue for the John-Nirenberg theorem for the dyadic model of the Dirichlet space (denoted $\mathcal{D}_a^d$), which I suspect to hold true for functions in B_a^p:

Theorem 3.1. *If f is the boundary value of a function that gives rise to a Carleson measure for $\mathcal{D}_a^d$ then there exist absolute constants A, B, such that for each dyadic interval I_0 and $\mu > 0$ we have that*

1. *If $a = 0$ then,*

$$|\{x \in I_0 : |f - m_{I_0}(f)| > \mu\}| \leq A|I_0| \exp(-\exp(B\mu^{p'})) \tag{3.2}$$

2. *If $0 < a < 1$ then,*

$$|\{x \in I_0 : |f - m_{I_0}(f)| > \mu\}| \leq A|I_0| \exp(-B\mu) \tag{3.3}$$

However, in contrast to the classical case of BMO, these conditions, which are satisfied by *all* bounded functions, cannot be a complete characterization of such f. I am optimistic and hope to find such a characterization.

References

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