Solutions - Homework 8

Assignment: Chapter 9, pp 212-214, #2, 5, 9, 10, 11, 16, 20

#2) Let \( f(x, y) = \left(\frac{x^2 + y^2}{2}\right)^{\alpha} \). For \( \alpha \leq \frac{1}{2} \),

\[
\frac{\partial f}{\partial x}(0, 0) \text{ does not exist so } f \text{ is not differentiable at } (0, 0).
\]

\( \text{For } \alpha > \frac{1}{2} \), \( f \) is continuously differentiable on \( \mathbb{R}^2 \) since

\[
\frac{\partial f}{\partial x}(x, y) = 2x \left(\frac{x^2 + y^2}{2}\right)^{\alpha-1}, \quad \frac{\partial f}{\partial y}(x, y) = 2y \left(\frac{x^2 + y^2}{2}\right)^{\alpha-1},
\]

are continuous on \( \mathbb{R}^2 - \{0, 0\} \), and

\[
\lim_{(x, y) \to (0, 0)} \frac{\partial f}{\partial x}(x, y) = \lim_{(x, y) \to (0, 0)} \frac{x}{x^2 + y^2} \left(\frac{x^2 + y^2}{2}\right)^{\alpha-\frac{1}{2}} = 0
\]

\[
= \lim_{x \to 0} \frac{(x^2)^{\alpha}}{x} = \frac{\partial f}{\partial x}(0, 0)
\]

with similar results for \( \frac{\partial f}{\partial y} \).

#5) Let \( f: \text{open } \subseteq \mathbb{R}^n \to \mathbb{R} \)

(6) If there are continuous functions \( A_1, \ldots, A_m \) on \( \mathbb{R}^n \) for which

\[
f(x, y) = \sum_{i=1}^{m} A_i(x, y) \left(x_i - y_i\right)
\]

it follows easily that \( \frac{\partial f}{\partial x_i}(x, y) \) exists and is equal to the continuous function \( A_i(x, x) \) for each \( i \), so \( f \) is continuously differentiable on \( \mathbb{R}^n \)

(6ii) When \( \text{U is convex, there is a natural and useful choice of continuous functions } A_i \) on \( \mathbb{R}^n \).
\[ A_i(x,y) = \int_0^1 \frac{\partial f}{\partial x_i}(y + t(x-y)) \, dt \quad \text{and} \quad f(x) - f(y) = \int_0^1 \frac{\partial f}{\partial x}(y + t(x-y)) \, dt \]

\[ = \sum_i \Lambda_i(x,y) \xi_i - y_i \] by the chain rule and an easy argument using uniform continuity of \( \frac{\partial f}{\partial x} \) on compact sets makes \( A_i \) continuous on \( U \times U \).

In general, for \( x \neq y \), \( \Lambda_i(x,y) = \frac{f(x) - f(y)}{||x-y||^2} \xi_i - y_i \)
gives \( \xi \) continuous functions \( \{ \Lambda_i(x,y) \} \in (U \times U \setminus \{x \neq y\}) \) satisfying \( \sum_i \Lambda_i(x,y) \xi_i - y_i \)

\[ \geq (f(x) - f(y)) \frac{\sum \xi_i - y_i}{\frac{1}{||x-y||^2}} = f(x) - f(y) \]

One can then construct a weighted average
\[
\tilde{f}(x,y) = \int_0^1 \frac{\partial f}{\partial x_i}(y + t(x-y)) \, dt \quad \text{for} \quad ||x-y|| \leq \varepsilon
\]
and \( \Lambda_i(x,y) = \frac{1}{||x-y||^2} \) to come up with \( A_i \)'s satisfying \( (*) \).

This is pointless and the details are unimportant since for a non-convex \( U \), there's no useful choice \( \xi \) \( A_i \)'s satisfying \( (*) \) and any particular choice \( \xi \) \( A_i \)'s can always be replaced by \( A_i + B_i \) for infinitely many choices \( \xi \) \( B_i \)'s satisfying \( \sum B_i(x,y) \xi_i - y_i \)

\[ = 0 \quad \forall x,y \] . Such \( B_i \)'s have nothing to do with \( f \).

The upshot is that this problem is badly phrased --- it should have specified \( U \) to be convex or asked only \( A_i \)'s continuous at points \( \xi \) with
a construction like this on p. 196, so

\[ G \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y} \right) \left( \frac{1}{\sqrt{y}} e^{-\frac{x^2}{4y}} \right) \]

\[ = e^{-\frac{x^2}{4y}} \left\{ \frac{1}{\sqrt{y}} \left( \frac{2x}{4y} \right)^2 - \frac{2}{\sqrt{y^3}} y - \frac{1}{2} \frac{1}{y^{3/2}} + \left( \frac{x^2}{4y^2} \right) \frac{1}{\sqrt{y}} \right\} = 0 \]

for \((x, y) \in \mathbb{R} \times [0, \infty)\). Then by Problem 7 and the Chain Rule

for each \(f : [a, b] \rightarrow \mathbb{R}\) continuous,

\[ G \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y} \right) \left( \frac{1}{\sqrt{y}} e^{-\frac{-(x-t)^2}{4y}} \right) dt \]

\[ = \int_a^b f(t) \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y} \right) \left( \frac{1}{\sqrt{y}} e^{-\frac{(x-t)^2}{4y}} \right) dt = 0 \]

\[ \downarrow \]

The notation \(\nabla \neq 0\) is non-standard. For \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(\|x\|^2 = \sum x_i^2\), the function \(G(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\|x-t\|^2/4t}\) is the fundamental solution \(u\) of the \(n\)-dim heat equation \((\nabla^2 - \frac{\partial}{\partial t})u = 0\), and, under mild hypotheses, one can prove that every solution \(u\) is generated by \(G\) via \(u(x, t) = \int_{\mathbb{R}^n} f(y) G(x-y, t) dy\) for \(f\) smooth. See books on mathematical physics for details.
Suppose \( f : (a, b) \times (c, d) \to \mathbb{R} \) is continuously differentiable and that \( \frac{\partial^2 f}{\partial x \partial y} \) exists and is 0 at each point. Then, for \( c < y < d \),

\[ f(x, y) \to \frac{\partial f}{\partial y} (x, y_0) \text{ is constant on } (a, b). \]

Let \( g(y) \) be the constant value.

For any choice \( y_0 \in (c, d) \), the Fundamental Theorem of Calculus gives

\[ f(x, y) - f(x, y_0) = \int_{y_0}^{y} \frac{\partial f}{\partial y} (x, u) \, du = \int_{y_0}^{y} g(u) \, du = f(x, y) \]

so

\[ f(x, y) = f_1(x) + f_2(y) \]

with \( f_1(x) = f(x, y_0) \).

Obviously, \( f_1 \) and \( f_2 \) are \( C^1 \) functions.

Caution: In general, when \( \frac{\partial^2 f}{\partial x \partial y} = 0 \) one cannot conclude that

\[ f(x, y) = f_1(x) + f_2(y) \text{ only on regions of the form } \]

\[ \bigcup_{x \in (a, b)} U = \bigcup_{y \in (c, d)} (x, y) : c(y) < x < b(y) \]

The proof is the same as above.

11. We suppose \( f = (f_1, \ldots, f_n) \) is a \( C^1 \) vector field on a ball \( U \subseteq \mathbb{R}^n \) with \( \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \) and wish to find \( \lambda \) for which \( F = \nabla \lambda \), i.e.

\[ \frac{\partial f_i}{\partial x_j} = \frac{\partial \lambda}{\partial x_i} \text{ for each } j \]

Let \( a = (a_1, \ldots, a_n) \) be the center of the ball.

As in the theorem on continuous partitions implying differentiability, for \( x = (x_1, \ldots, x_n) \in U \), we let

\[ P_i = (x_1, \ldots, x_{i-1}, x_i + \delta, x_{i+1}, \ldots, x_n) \text{ for } 0 \leq i \leq n \]

and parametrize the line segment joining \( P_{i-1} \) and \( P_i \).
by $G_i(x) = (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$, \( 0 \leq t \leq x_i 
abla x_i(\vec{x}) \), \( 0 \leq t \leq x_i \)

Define $F(x) = \sum_{c=1}^{n} \int_{a_c}^{x} \frac{P_c}{\xi} \left( \frac{\partial}{\partial \xi} G_i(\xi) \right) \, d\xi$

Then $\frac{\partial F}{\partial x_j} = \overline{f_j}(P_j) + \sum_{i=1}^{n} \int_{a_i}^{x_i} \frac{\partial P_i}{\partial x_j} \left( G_i(\xi) \right) \, d\xi$

\[= \overline{f_j}(P_j) + \sum_{i=1}^{n} \int_{a_i}^{x_i} \frac{\partial f_j}{\partial x_i} \left( G_i(\xi) \right) \, d\xi \]

(Fund Thm of Calc) \[= \overline{f_j}(P_j) + \sum_{i=1}^{n} \overline{f_j}(P_i) - \overline{f_j}(P_{i-1}) \]

\[= \overline{f_j}(P_n) = \overline{f_j}(x) \]

(with more effort, a variation of this proof works for \( U \) any convex open set)

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\( (ii) \) Let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a $C^2$ function

with $a = (a_1, \ldots, a_n)$ a critical point for $f$.

i.e. $\frac{\partial f}{\partial \xi_i}(a) = 0$ for $1 \leq i \leq n$. Then the 2nd order Taylor expansion for $f$ about $a$ gives

$f(x) - f(a) = \frac{1}{2} (x-a)^t H(x-a) + \|x-a\|^2 E(x)$

where $E(x) \to 0$ as $x \to a$ and $H$

= Hessian matrix of $f$ at $a$ - non-symmetric matrix with $i,j$ entry \( \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} (a) \)

Let $m_1$ and $m_2$ be the minimum and maximum values of the continuous function

$v \mapsto v^t H v$ on the compact set

$S = \{ v \in \mathbb{R}^n : \|v\| = 1 \}$. Choose $c_0 > 0$ for which

$|E(x)| < \frac{1}{2} \min \{ |m_1|, |m_2| \}$ when $\|x-a\| < c_0$ (assuming $m_1$ and $m_2$ are non-zero).
Then, for \( \|x-a\| < r_0 \),

\[
\left( \frac{1}{2} m_1 + E(\varepsilon) \right) \|x-a\|^2 \leq \frac{1}{2} m_2 + E(\varepsilon) \|x-a\|^2
\]

Case: (i) \( m_2 > m_1 > 0 \) \( \text{ (H positive definite) } \)

\[
\Rightarrow \ f(x) - f(a) > 0 \quad \text{for} \ 0 < \|x-a\| < r_0
\]

\[
\Rightarrow \ f \text{ has a local minimum at } a
\]

(ii) \( m_1 \leq m_2 < 0 \) \( \text{ (H negative definite) } \)

\[
\Rightarrow \ f(x) - f(a) < 0 \quad \text{for} \ 0 < \|x-a\| < r_0
\]

\[
\Rightarrow \ f \text{ has a local maximum at } a
\]

(iii) \( m_1 < 0 < m_2 \) \( \text{ (H neither positive semi-definite nor negative semi-definite) } \)

Hello u_1 \text{ is a unit vector for which } u_1^T Hu_1 = m_1.
and for \( 0 < t < r_0 \),

\[
\Rightarrow \ f(a) > f(a + tu_1)
\]

so \( f \) has neither a local maximum nor a local minimum at \( a \). (\( a \) is a saddle point for \( f \))

(iv) \( m_1 = m_2 = 0 \). Then the test fails —

\[ H \] doesn't detect whether or not \( f \) has a local max/min at \( a \).

(20) We assume that \( S \) is a closed convex set in \( \mathbb{R}^n \) with \( f \) a \( C^1 \) function from

an open set containing \( S \) into \( \mathbb{R}^n \) and

\[ f(S) \subseteq S \]. Then, for \( x, y \) in \( S \)

\[
f(y) - f(x) = \int_0^1 \frac{d}{dt} f(x + t(y-x)) \ dt
\]

\[
= \int_0^1 (df)_{x+t(y-x)} \ dt
\]

As we'll show later, for any norm \( \| \cdot \| \) on \( \mathbb{R}^n \), we have
\[ \|f(y) - f(x)\| \leq \int_0^1 \|A(t)\|_{\mathcal{F}} \, dt \quad \text{for each } x, y \in \mathbb{R} \]

Let each \( Z, \|A(t)\|_{\mathcal{F}} \leq C_2 \|v\| \) on the compact set \( S = \{ v : \|v\| = 1 \} \). Hence, for \( c = \max_{v \in S} C_2 \|v\| \)

\[ \|f(y) - f(x)\| \leq \int_0^1 c \|y-x\| \, dt = c \|y-x\| \]

when \( c \leq 1 \) \( f \) is a contraction mapping on the complete metric space \( (S, \|\cdot\|) \)

(i) Using \( \|v\| = \|v\|_\infty \) \[ \|A(t)\|_{\mathcal{F}} = \max_{v \in \mathcal{F}} \|A(t)\|_v \leq C_2 \|v\|_\infty \]

\[ \max_{v \in \mathcal{F}} \left( \sum_{d=1}^n \frac{\partial f_i}{\partial x_j}(v) \right) \|v\|_\infty \]

\[ \leq \left( \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(v) \right) \|v\|_\infty \]

So we get a contraction mapping when \( \sum_{j=1}^n \|\frac{\partial f_i}{\partial x_j}\|_\infty \|v\|_\infty \leq 1 \) \( \forall i \)

(ii) Using \( \|v\| = \|v\|_2 \) \( \leq \|v\|_\infty \)

\[ \|A(t)\|_{\mathcal{F}} \leq \sum_{j=1}^n \|\frac{\partial f_i}{\partial x_j}(v)\|_2 \leq \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(v) \|v\|_2 \]

\[ \leq \max_{v \in \mathcal{F}} \left( \sum_{j=1}^n \left( \frac{\partial f_i}{\partial x_j}(v) \right)^2 \right) \|v\|_2 \]

(iii) Using \( \|v\|_2 = \|v\|_\infty \) and Cauchy-Schwarz

\[ \|A(t)\|_{\mathcal{F}}^2 \leq \sum_{j=1}^n \left( \sum_{d=1}^n \frac{\partial f_i}{\partial x_j}(v) \right)^2 \|v\|_2 \]

So we get a contraction mapping

\[ \sum_{j=1}^n \left( \frac{\partial f_i}{\partial x_j}(v) \right)^2 \leq \epsilon^2 < 1 \forall v \]