The Foliated Liouville Problem

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November 6, 2004 - EIU
General setting – harmonic version

- $M$ – a compact connected manifold without boundary;

- $\mathcal{F}$ – a continuous foliation of $M$ by (smooth) Riemannian manifolds.

**Definition:** The foliated space $(M, \mathcal{F})$ has the *Liouville property* if continuous leafwise harmonic functions on $M$ are leafwise constant. If this property holds, we also say that $(M, \mathcal{F})$ is *harmonically simple*.

**Problem:** Characterize the $(M, \mathcal{F})$ that have this property.
Holomorphic version

- $M$ – a compact connected manifold without boundary;

- $\mathcal{F}$ – a continuous foliation of $M$ by complex manifolds.

**Definition:** The foliated space $(M, \mathcal{F})$ is *holomorphically simple* if continuous leafwise holomorphic functions on $M$ are leafwise constant.
Uninteresting examples:

- If the leaves of $\mathcal{F}$, individually, do not admit bounded harmonic functions, then $(M, \mathcal{F})$ is harmonically simple. This is the case, for example, if the leaves of $\mathcal{F}$ have non-negative Ricci curvature. (S.-T. Yau: If $L$ is a complete Riemannian manifold with non-negative Ricci curvature, then there are no non-constant bounded harmonic function.)

- In the holomorphic case, if each leaf of $\mathcal{F}$ admits $\mathbb{C}^m$ as a (holomorphic) covering space, then $(M, \mathcal{F})$ is holomorphically simple.

- If the leaves of $\mathcal{F}$ are holomorphically parallelizable (e.g., $\mathcal{F}$ is the orbit foliation of a locally free action of a complex Lie group on $M$), then $(M, \mathcal{F})$ is holomorphically simple.
Some “negative” results
(conditions that imply the Liouville property)

**Theorem 1.** (Holomorphic) *If the closure of each leaf of \((M, \mathcal{F})\) contains (at most) countably many minimal sets. Then the foliation is holomorphically simple.*

**Theorem 2.** (Holomorphic) *If \((M, \mathcal{F})\) has codimension-one, then it is holomorphically simple.*

**Theorem 3.** [Garnett] (Harmonic) *If the union of the supports of harmonic measures is all of \(M\), then \((M, \mathcal{F})\) is harmonically simple.*
Foliated bundle over $S$ with fiber $X$

Let $S'$ be a compact connected Riemannian (complex, in the holomorphic case) manifold, $\tilde{S}$ its universal covering space, and $\gamma$ the fundamental group of $S$ represented as the group of deck transformations of $\tilde{S}$. Let $X$ be a compact connected space on which $\Gamma$ acts by homeomorphisms. Let $M = (\tilde{S} \times X)/\Gamma$ denote the space of orbits for the action of $\Gamma$ on $\tilde{S} \times X$ defined by $(s, x) \cdot \gamma := (s\gamma, \gamma^{-1}(x))$. Then $M$ is foliated by Riemannian (complex) manifolds locally isomorphic to $S$.

If $\rho$ denotes the action of $\Gamma$ on $X$, the associated foliated bundle will be written $(M_\rho, F_\rho)$.

Of particular interest: $S$ is a Riemann surface of genus $g \geq 2$. 
“Negative” results for foliated bundles

Let $S$ be a compact Riemann surface of genus at least 2. If $G$ is an algebraic group, let $\text{Hom}(\Gamma, G)$ denote the variety of homomorphisms from $\Gamma$ into $G$.

**Theorem 4.** Let $(M_\rho, F_\rho)$ be the foliated bundle over $S$ with fiber $P^{n-1}(\mathbb{C})$ and action induced by $\rho : \Gamma \rightarrow GL(n, \mathbb{C})$. Then there is a Zariski open dense subset $U$ in $\text{Hom}(\Gamma, GL(n, \mathbb{C}))$ such that, for each $\rho \in U$, $(M_\rho, F_\rho)$ is both holomorphically and harmonically simple.

**Theorem 5.** Let $\Lambda$ be a Gromov-hyperbolic group, $X$ the boundary of $\Lambda$, and $S$ a compact connected Riemannian manifold with fundamental group $\Gamma$. Suppose that $\Gamma$ acts on $X$ via a homomorphism $\rho : \Gamma \rightarrow \Lambda$ and let $(M_\rho, F_\rho)$ be the corresponding foliated bundle over $S$. Then $(M_\rho, F_\rho)$ is harmonically simple. The same holds if $\Lambda$ is replaced by $SL(2, \mathbb{C})$. 
"Positive" results

**Theorem 6.** (Holomorphic) *There exists a compact real analytic foliation $(M, \mathcal{F})$, a foliated bundle over a compact Riemann surface, and a real analytic leafwise holomorphic function on $M$ that is not leafwise constant.*

- $\mathbb{D}$ – unit disc; $\Gamma$ a cocompact lattice in $SU(1, 1)$; $S = \mathbb{D}/\Gamma$;

- $M = (\mathbb{D} \times C)/\Gamma$; $C = \{ [z_1, z_2, t] \in P^4(\mathbb{R}) : |z_1|^2 - |z_2|^2 = t^2 \}$;

- $\left( \begin{array}{cc} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{array} \right) \cdot [z_1, z_2, t] = [\alpha z_1 + \beta \bar{z}_2, \alpha z_2 + \beta \bar{z}_1, t]$;

- $f(z, [\alpha, \beta, t]) := \frac{\bar{\alpha}z - \beta}{-\beta z + \alpha}$. $f$ is $\Gamma$-invariant.
Harmonically non-simple, holomorphically simple example

\[ S = \mathbb{D}/\Gamma; \ \Gamma \subset SU(1,1) \text{ a cocompact lattice.} \]

**Theorem 7.** \( \exists (M,F), \text{ a fol. bund. over } S \text{ with fiber } S^2, \text{ such that:} \)

- \((M,F)\) is \(C^\omega\) on complement of pair of leaves, \(S_1, S_2, \text{ homeom. to } S; \)

- \((M,F)\) is ergodic with respect to the smooth measure class;

- \((S_1 \cup S_2)^c\) has a \(C^\omega\) compactification, which is an ergodic foliated bundle over \(S\) with fiber \(S^1 \times [0, 2\pi]\);

- For both \((M,F)\) and the above analytic compactification, the Liouville property does not hold. A continuous, leafwise harmonic, not leafwise constant, can be found that is real analytic on the complement of \(S_1 \cup S_2.\)
A “universal” non-Liouville foliation

• \( X_0 = \text{Har}(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{C} \text{ harmonic, } |f(z)| \leq 1 \}; \)

• \( PSU(1, 1) \) acts on \( X_0 \) by \((g, f) \mapsto f \circ g^{-1}; \)

• \( \Gamma \subset PSU(1, 1) \) a cocompact lattice; \( M_0 = (\mathbb{D} \times X_0)/\Gamma; \)

• \( \Phi([z, f]) := f(z), \Phi : M_0 \to \mathbb{C}. \)

Finite dimensional examples are obtained by looking for finite dimensional closed orbits of \( PSU(1, 1) \) on \( X_0 \), then restricting \( \Phi \) to the foliated subspace of \( M_0 \) associated to that orbit.
Dynamics of subgroups of $PSU(1, 1)$ acting on $Har(\mathbb{D})$

Devaney: A continuous map $f : X \rightarrow X$ of a metric space $X$ generates a chaotic dynamical system if:

- There exists a dense orbit (topological transitivity);

- The set of periodic points (finite orbits) is dense;

- Sensitive dependence on initial conditions. (There exists $\delta > 0$ such that for all $x \in X$ and every neighborhood $N$ of $x$, there exists $y \in N$ and positive integer $n$ such that $f^n(x)$ and $f^n(y)$ are more than $\delta$ apart.)

**Theorem 8.** Let $\gamma$ be a hyperbolic or parabolic element of $PSL(2, \mathbb{R})$, regarded as a transformation on $Har(\mathbb{D})$. Then $\gamma$ defines a chaotic dynamical system.
Questions

• If \((M, \mathcal{F})\) has codimension 2, is it holomorphically simple?

• If \((M, \mathcal{F})\) has codimension 1, is it harmonically simple?

• Clarify relationship of holom. simple and harm. simple foliations. (Note: if \(H^1_{dR}(M, \mathcal{F}) = 0\), harm. simple \(\Leftrightarrow\) holom. simple.)

• Let \((M, \mathcal{F})\) be a compact foliated bundle over \(\Gamma \backslash G/K\), where \(G/K\) is an irreducible locally symmetric space of rank at least two. Show (or give counter-example) that the foliation is harmonically simple.

• Given a hyperbolic and a parabolic element in \(PSL(2, \mathbb{R})\), are the dynamical systems they define on \(Har(\mathbb{D})\) topologically equivalent?