EFFECT OF EROs ON DETERMINANTS

Let $A$ be a square matrix:

1) if a multiple of one row of $A$ is added to another to get a matrix $B$, then $\det A = \det B$
   \textit{(row replacement has no effect on determinant)}

2) If two rows of $A$ are interchanged to get $B$, then $\det B = -\det A$
   \textit{(each row swap reverses the sign of the determinant)}

3) If one row of $A$ is multiplied by $k$ ($k \neq 0$) to get $B$, then $\det B = k\det A$
   \textit{(rescaling a row by a factor of $k$ also rescales the determinant by a factor of $k$)}

Example: $\det \begin{bmatrix} 5a & 5b \\ c & d \end{bmatrix} = 5 \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

\textit{(3) often used to “factor out” a number from a row}

Why?
Rephrase in terms of elementary matrices

**Theorem**  
Suppose $A$ is $n \times n$ and $E$ is an $n \times n$ elementary matrix

then

1. $\det(EA) = \det(E)\det(A)$

and

2. $\det(E) = \begin{cases} 1 & \text{if } E \text{ is: add multiple of row to another} \\ -1 & \text{if } E \text{ is: interchange two rows} \\ k & \text{if } E \text{ is: rescale a row by a factor of } k \end{cases}$

__________________ WHY?______________________________

Check this first for $2 \times 2$ matrices

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $2 \times 2$ and that $E$ is a $2 \times 2$ elementary matrix ($\leftrightarrow$ ERO)

1. If $E$ represents a row replacement (say, $k$*row 1 added to row 2)

$EA = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \cdot A = \begin{bmatrix} a & b \\ ka + c & kb + d \end{bmatrix}$,

$\det(E) = 1$

so

$\det(EA) = kab + ad - kab - bc = ad - bc$

$= \det(A)$  (so this ERO doesn't change $\det(A)$)

$= \det(E) \cdot \det(A)$

$\uparrow$

$1$
2. If $E$ represents a row rescaling (say, a rescaling of row 1)

$$EA = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \cdot A = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

$\det(E) = k$

so

$$\det(EA) = k(ad - bc) = k \det(A)$$

(\textit{so this ERO multiplies} $\det(A)$ \textit{by} $k$)

$$= \det(E) \cdot \det(A)$$

3. If $E$ represents a row interchange

$$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot A = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$\det(E) = -1$

so

$$\det(EA) = (bc - ad) = (-1)\det(A)$$

(\textit{so this ERO changes sign of} $\det(A)$)

$$= \det(E) \cdot \det(A)$$

So, by direct checking we find that the \textit{Theorem is true for} $2 \times 2$ \textit{matrices}.

Next argue

Theorem true for $2 \times 2$ matrices:

forces the theorem to be true for $3 \times 3$ matrices;

Then

Theorem true for $3 \times 3$ matrices:

forces the theorem to be true for $4 \times 4$ matrices;

And in general
Theorem true for matrices forces the theorem to be true for \((n + 1) \times (n + 1)\) matrices.

This is called a proof by induction: we argue “up the ladder”; it's true for \(2 \times 2\) matrices, and whenever it's true for one size, \(n \times n\), then it must also be true for the “next size up”, \((n + 1) \times (n + 1)\). This means it must be true for all sizes of square matrices.

We didn’t do this general argument. But illustrated by showing why, for a row replacement ERO, “true for \(2 \times 2\)” forces “true for \(3 \times 3\)”. The same style argument is how you argue from size \(n \times n\) to size \(n + 1 \times (n + 1)\).

To illustrate: Why does the fact that

\[(**)\]

\[\det(EA) = \det(E)\det(A);\]

force it also to be true for \(3 \times 3\)?

For Row replacement:

\[\det(E) = 1\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
k & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
= \begin{bmatrix}
a & b & c \\
d + ka & e + kb & f + kc \\
g & h & i
\end{bmatrix}
\]

expand along a row that’s “uninvolved” with the ERO

\[\det(EA) = g \det\begin{bmatrix}
b & c \\
e + kb & f + kc
\end{bmatrix}
= -h \det\begin{bmatrix}
a & c \\
d + ka & f + kc
\end{bmatrix}
= +i \det\begin{bmatrix}
a & b \\
d + ka & e + kb
\end{bmatrix}\]

why? because, without even evaluating the determinants, we know this row operation does not affect the determinant for \(2 \times 2\) matrices.

\[
\begin{align*}
&= g \det\begin{bmatrix}
b & c \\
e & f
\end{bmatrix}
- h \det\begin{bmatrix}
a & c \\
d & f
\end{bmatrix}
+ i \det\begin{bmatrix}
a & b \\
d & e
\end{bmatrix} \\
&= \det(A) = 1 \cdot \det(A) \\
&= \det(E) \cdot \det(A)
\end{align*}
\]
We then proved in class (see textbook):

If $E_1, \ldots, E_p$ are $n \times n$ elementary, and $A, B$ are $n \times n$:

1) $\det(E_p \cdot \ldots \cdot E_1 A) = \det(E_p) \cdot \ldots \cdot \det(E_1) \cdot \det(A)$
2) $\det(E_p \cdot \ldots \cdot E_1) = \det(E_p) \cdot \ldots \cdot \det(E_1)$ (determinant of product = product of
determinants, for elementary matrices)
3) $\det(AB) = \det(A) \cdot \det(B)$ for any two $n \times n$ matrices $A, B$
   and this easily generalizes to:
   determinant of product = product of determinants for any number of square matrices.

(If you know 3), then 1) and 2) are of course automatically true; but we "inched up" to 3) by proving 1) and 2) first.)
**Theorem** For an \( n \times n \) matrix \( A \), \( \det A^T = \det A \)

**Proof** True if \( A \) is \( 2 \times 2 \) (why?)

\[ \text{just check: is } \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \]?

The proof is again “by induction”: here’s why “it works for \( 2 \times 2 \)” forces that “is works for \( 3 \times 3 \).” Why does this make it true when \( A \) is \( 3 \times 3 \)?

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \]

First row \( A \) = first column \( A^T \); use this row & column to compute the determinants

\[ \det A=\text{sum of } \downarrow \quad \det A^T=\text{sum of } \downarrow \]

\[ (\text{across row } 1) \quad (\text{down column } 1) \]

\[ a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \quad a_{11} \begin{bmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{bmatrix} \]

\[ -a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \quad -a_{12} \begin{bmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{bmatrix} \]

\[ +a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad +a_{13} \begin{bmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{bmatrix} \]

colored pairs have same determinant because they are \( 2 \times 2 \) and transposes of each other

So \( \det A = \det A^T \) when \( A \) is \( 3 \times 3 \).

To show that, then \( \det A = \det A^T \) must also be true for \( 4 \times 4 \), then that it must also be true for \( 5 \times 5 \), etc. uses exactly the same style calculation.