We start with an “empty” matrix $P$ that is $2008 \times 2008$

John and Mary play the following game:

John picks any number and enters it somewhere in the matrix.

Then Mary picks a number and puts it somewhere in the matrix.

Then John enters another number, then Mary, etc., taking turns back and forth until the matrix is filled.

The det $G$ is computed. John wins the game if $\det G \neq 0$; Mary wins if $\det G = 0$.

Does either payer have a winning strategy — that is, an algorithm for making plays that will guarantee a win?
Effect of a Linear Transformation on Area

Theorem Suppose \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is a linear transformation with standard matrix \( A \).

1) Let \( S \) be a parallelogram in \( \mathbb{R}^2 \). The image of the parallelogram is \( T(S) = \{ T(\mathbf{x}) : \mathbf{x} \text{ in } S \} \) and another parallelogram (possibly “collapsed” to line segment or point)

\[
\text{(area of } T(S) \text{)} = |\det A| \cdot (\text{area of } S)
\]

2) If \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is linear with standard matrix \( A \). Let \( S \) be a parallelepiped in \( \mathbb{R}^3 \). Then

\[
\text{(volume of } T(S) \text{)} = |\det A| \cdot (\text{volume of } S)
\]
And not just for parallelograms!

“Theorem”  For linear $T : \mathbb{R}^2 \to \mathbb{R}^2$, or for $T : \mathbb{R}^3 \to \mathbb{R}^3$

1) if $S$ is any subset of the plane that has a finite area, then

$$\text{area} \ (T(S)) = |\det A| \cdot \text{area}(S)$$

2) if $S$ is any subset of the $\mathbb{R}^3$ that has a finite volume, then

$$\text{volume} \ (T(S)) = |\det A| \cdot \text{volume}(S)$$
$\mathbb{R}^n$ is an example of what is called a **vector space**.

$\mathbb{R}^n$ contains **vectors** like $u = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

and has **two operations**

i) **addition** (+)

ii) **multiplication by a scalar** that obey these rules:

\[
\begin{array}{c}
\text{For all } \quad u, v, w \in V \quad \text{(vectors)} \\
\text{and for all } \quad c, d \in \mathbb{R} \quad \text{(scalars)} \\
1. \quad u + v \text{ is in } V \quad \text{(closure under addition)} \\
2. \quad u + v = v + u \\
3. \quad (u + v) + w = u + (v + w) \\
4. \quad \text{There is a vector (denoted } 0 \text{) with the property } u + 0 = u \\
5. \quad \text{For each vector } u, \text{ there is another vector } \quad \text{called } -u \text{ for which } u + (-u) = 0 \\
6. \quad cu \text{ is in } V \quad \text{(closure under scalar multiplication)} \\
7. \quad c(u + v) = cu + cv \\
8. \quad (c + d)u = cu + du \\
9. \quad c(du) = (cd)u \\
10. \quad 1u = u
\end{array}
\]
A vector space $V$ is a nonempty set of objects and two operations which we call

i) addition $(+)$
ii) multiplication by a scalar that obey these rules ("axioms for a vector space"):

For all $u, v, w \in V$ (vectors) and for all $c, d \in \mathbb{R}$ (scalars)

1. $u + v$ is in $V$ (closure under addition)
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a vector (denoted 0) with the property $u + 0 = u$
5. For each vector $u$, there is another vector called $-u$ for which $u + (-u) = 0$
6. $cu$ is in $V$ (closure under scalar multiplication)
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c(du) = (cd)u$
10. $1u = u$
From these axioms for a vector space A1-A10, we can deduce additional properties that are true in every vector space, such as:

for every vector \( \mathbf{u} \), \( 0 \cdot \mathbf{u} = \mathbf{0} \)

because

\[
0 \cdot \mathbf{u} = (0 + 0) \cdot \mathbf{u} = 0 \cdot \mathbf{u} + 0 \cdot \mathbf{u}
\]

Add \(- (0 \cdot \mathbf{u})\) to both sides

\[
0 \cdot \mathbf{u} - (0 \cdot \mathbf{u}) = (0 \cdot \mathbf{u} + 0 \cdot \mathbf{u}) - (0 \cdot \mathbf{u})
\]

\[
\mathbf{0} = 0 \cdot \mathbf{u} + (0 \cdot \mathbf{u} - (0 \cdot \mathbf{u}))
\]

\[
0 = 0 \cdot \mathbf{u} + \mathbf{0} = 0 \cdot \mathbf{u}
\]

and get

\[
\mathbf{0} = 0 \cdot \mathbf{u}
\]
Suppose \( V \) is a vector space and that \( H \) is a subset of \( V \) (written \( H \subseteq V \)).

\( H \) is called a sub\textit{space} of \( V \) if

i) \( 0 \in H \)

ii) if \( u, v \in H \), then \( u + v \in H \)

\( (H \) is “closed under addition”)

iii) if \( u \in H \) and \( c \) is a scalar, then \( cu \in H \)

\( (H \) is “closed under scalar multiplication”)

All other properties needed for a vector space are satisfied by vectors in \( H \) automatically since they are true already in the given vector space \( V \):

for example, if \( u, v \in H \), then \( u + v = v + u \)

because \( u, v \) are also in the larger set, and \textit{we already know} that \( u + v = v + u \) is true in vector space \( V \).

Think of a subspace \( H \) as a smaller “self-contained” vector space that lives inside the larger vector space \( V \).
Theorem

If \[ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \] are vectors in \( V \), then

Let \( H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \)

\[ = \{\text{all possible linear combinations of } \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \]

\( H \) is a subspace of \( V \). Why?

i) \( \mathbf{0} = 0\mathbf{v}_1 + \ldots + 0\mathbf{v}_p \) is in \( H \)

ii) If \( \mathbf{u}, \mathbf{v} \in H \), that means

\[ \mathbf{u} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p \] for some weights \( c_1, \ldots, c_p \) and

\[ \mathbf{v} = d_1\mathbf{v}_1 + \ldots + d_p\mathbf{v}_p \] for some weights \( d_1, \ldots, d_p \)

Then

\[ \mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + \ldots + (c_p + d_p)\mathbf{v}_p \]

so \( \mathbf{u} + \mathbf{v} \) is in \( H \), and

iii) \( c\mathbf{u} = (cc_1)\mathbf{v}_1 + \ldots + (cc_p)\mathbf{v}_p \) is in \( H \)
Example

Suppose $A$ is an $m \times n$ matrix.

Let $H = \{x \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} =$ the “solution set” of the homogeneous equation $A\mathbf{x} = \mathbf{0}$

$H$ is a subspace of $\mathbb{R}^n$ because

- **$\mathbf{0}$ is in $H$** (because $A\mathbf{0} = \mathbf{0}$)

- if $\mathbf{u}$ and $\mathbf{v}$ are in $H$

  then $\mathbf{u} + \mathbf{v}$ is in $H$ (because $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$)

and

- if $\mathbf{u}$ in $H$ and $c$ is a scalar

  then $c\mathbf{u}$ is in $H$ (because $A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{0} = \mathbf{0}$)

$H$ is called the null space of the matrix $A$. 
Practice

1) Let \( \mathbb{P} \) be the vector space of all polynomials

a) What is the “zero vector” in this space?

b) Is the “zero vector” in Span \( \{1, t, t^2, t^3\} \)?
Why?

c) Consider a linear combination of these “vectors”

\[ 1, t, t^2, t^3 \]

that adds up to the zero vector:

\[ c_0 + c_1 t + c_2 t^2 + c_3 t^3 = 0 \]

What can you say about the weights \( c_0, \ldots, c_3 \)?
Why?