Review

A **vector space** \( V \) is a nonempty set of **objects** (which we call **vectors**) and **two operations** which we call

i) **addition** \((+\)**
ii) **multiplication by a scalar** that obey these **rules** (“the **axioms for a vector space**”):

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Description</th>
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<tbody>
<tr>
<td>1. ( u + v ) is in ( V )</td>
<td><strong>closure under addition</strong></td>
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<tr>
<td>2. ( u + v = v + u )</td>
<td><strong>associativity</strong></td>
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<td>3. ( (u + v) + w = u + (v + w) )</td>
<td><strong>distributivity</strong></td>
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<td>4. There is a vector (denoted ( 0 )) with the property ( u + 0 = u )</td>
<td><strong>existence of an identity element</strong></td>
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<td>5. For each vector ( u ), there is another vector ( -u ) for which ( u + (-u) = 0 )</td>
<td><strong>existence of an inverse</strong></td>
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<td>6. ( cu ) is in ( V )</td>
<td><strong>closure under scalar multiplication</strong></td>
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<tr>
<td>7. ( c(u + v) = cu + cv )</td>
<td><strong>distributivity</strong></td>
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<tr>
<td>8. ( (c + d)u = cu + du )</td>
<td><strong>linearity</strong></td>
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<td>9. ( c(du) = (cd)u )</td>
<td><strong>homogeneity</strong></td>
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<td>10. ( 1u = u )</td>
<td><strong>multiplicative identity</strong></td>
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Suppose \( V \) is a vector space and that \( H \) is a subset of \( V \) (written \( H \subseteq V \))

\( H \) is called a **subspace** of \( V \) if

i) \( 0 \in H \)

ii) if \( u, v \in H \), then \( u + v \in H \) (\( H \) is “closed under addition”)

iii) if \( u \in H \) and \( c \) is a scalar, then \( cu \in H \) (\( H \) is “closed under scalar multiplication”)

So a **subspace** of \( V \) is a special kind of subset: one that satisfies i), ii), and iii)
Theorem  Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \) be vectors in \( V \).

Then \( H = \text{span} \left\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \right\} \)

\[ = \left\{ \text{all possible linear combinations of } \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \right\} \]

is a subspace of \( V \).

Why?

i) \( \mathbf{0} = 0\mathbf{v}_1 + \ldots + 0\mathbf{v}_p \) is in \( H \)

ii) If \( \mathbf{u}, \mathbf{v} \in H \), that means

\[ \mathbf{u} = c_1 \mathbf{v}_1 + \ldots + c_p \mathbf{v}_p \text{ for some weights } c_1, \ldots, c_p \text{ and} \]

\[ \mathbf{v} = d_1 \mathbf{v}_1 + \ldots + d_p \mathbf{v}_p \text{ for some weights } d_1, \ldots, d_p \]

Then

\[ \mathbf{u} + \mathbf{v} = (c_1 + d_1) \mathbf{v}_1 + \ldots + (c_p + d_p) \mathbf{v}_p \text{ (a linear combination of }\mathbf{v}_1, \ldots, \mathbf{v}_p) \]

so \( \mathbf{u} + \mathbf{v} \) is in \( H \)

and

iii) \( c\mathbf{u} = (cc_1) \mathbf{v}_1 + \ldots + (cc_p) \mathbf{v}_p \) (a linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_p \))

so \( c\mathbf{u} \) is in \( H \).
Example

The set of all vectors \( H = \left\{ \begin{bmatrix} x + y + 2z \\ x - 3z \\ y + 7z \end{bmatrix} : x, y, z \text{ real} \right\} \) is a subspace of \( \mathbb{R}^2 \)

because we can write it as the space of a set of vectors:

\[
\begin{bmatrix}
x + y + 2z \\
x - 3z \\
y + 7z
\end{bmatrix}
= x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}
\]

\( H = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} \right\} \)

Example

Consider three vector spaces

\( \mathbb{P} = \) the set of all polynomials

\( \mathcal{D} = \{ f : f \text{ is a differentiable function with domain } (-\infty, \infty) \} \)

\( \mathcal{C} = \{ f : f \text{ is a continuous function with domain } (-\infty, \infty) \} \)

\( \mathbb{P} \) is a subspace of \( \mathcal{D} \), and \( \mathcal{D} \) is a subspace of \( \mathcal{C} \)

1) Name a few more subspaces of \( \mathcal{D} \)

2) What is a vector (function) in \( \mathcal{C} \) that is not in \( \mathcal{D} \) ?

3) Between any two vector spaces \( V \) and \( W \)

a linear transformation \( T : V \rightarrow W \) is a function that satisfies

a) for all vectors \( \mathbf{x}, \mathbf{y} \) in \( V \): \( T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \)

b) for all vectors \( \mathbf{x} \) in \( V \) and scalars \( c \): \( T(c \mathbf{x}) = c T(\mathbf{x}) \)

4) What is a familiar example of a linear transformation \( T : \mathbb{P} \rightarrow \mathbb{P} \) ?

\( T : \mathcal{C} \rightarrow \mathcal{D} \) ?
Two subspaces associated with a matrix $A$

Suppose $A$ is an $m \times n$ matrix. We can think of a transformation

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ where } T(x) = Ax$$

Example 1 Let $H = \{x \in \mathbb{R}^n : Ax = 0\}$ = the “solution set” of the homogeneous equation $Ax = 0$

$H$ is a subspace of $\mathbb{R}^n$ because

- $0$ is in $H$ (because $A0 = 0$)
- if $u$ and $v$ are in $H$
  - then $u + v$ is in $H$ (because $A(u + v) = Au + Av = 0 + 0 = 0$)
- and if $u$ is in $H$ and $c$ is a scalar
  - then $cu$ is in $H$ (because $A(cu) = cAu = c0 = 0$)

$H$ is called the null space of the matrix $A$, denoted $\text{Nul}(A)$

$H$ is also called the kernel of the linear transformation $T$, denoted $\text{ker}(T)$

$\text{Nul}(A)$ and $\text{ker}(T)$ are two different names for the same subspace. The term “null space” is more likely to be used when speaking of a matrix $A$, and the term “kernel” more likely when speaker of the linear transformation $T$.

$$\text{Nul}(A) = \text{ker}(T) \text{ is a subspace of } \mathbb{R}^n$$

(a subspace of the domain of the mapping $T(x) = Ax$)
Example 2  The columns of this same matrix $m \times n$ matrix $A$ are vectors in $\mathbb{R}^m$ and we can write $A = [a_1 \ a_2 \ \cdots \ a_n]$

Then $\text{Span} \ \{a_1, \ a_2, \ \cdots, \ a_n\}$ is a subspace of $\mathbb{R}^m$ (because is=t's the span of a set of vectors); it is called the column space of $A$ denoted $\text{Col}(A)$.

A vector $b$ in $\mathbb{R}^m$ is in the column space if and only if there is an $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ so that

$$b = x_1a_1 + x_2a_2 + \ldots + x_na_n = Ax = T(x)$$

So $b$ is in $\text{Col}(A)$ if and only if $b$ is in the range of the transformation $T$.

$$\text{Col}(A) = \text{range}(T)$$

(a subspace of the codomain of the linear transformation $T$)

Various computations illustrated:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 1 & 8 \\ -1 & -2 & 1 & 2 \\ 2 & 4 & 0 & 4 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

Here, $A$ is $4 \times 4$ so both $\text{Nul} \ A$ and $\text{Col} \ A$ are subspaces of $\mathbb{R}^4$. (In general, if $A$ is $m \times n$, and $n \neq m$, then $\text{Nul} \ A$ and $\text{Col} \ A$ are subspaces of two different vector spaces: $\text{Nul} \ A$ is a subspace of $\mathbb{R}^n$ and $\text{Col} \ A$ is a subspace of $\mathbb{R}^m$.)

1)  Is $\begin{bmatrix} 1 \\ 7 \\ 9 \\ 3 \end{bmatrix}$ in $\text{Nul} \ A$? No, $A \begin{bmatrix} 1 \\ 7 \\ 9 \\ 3 \end{bmatrix} \neq \mathbf{0}$ (multiply and check!)

2)  Give an explicit description of the vectors in $\text{Nul} \ A$. Solve $Ax = \mathbf{0}$ and write the solution in parametric vector form. From $\text{rref}(A)$ given above, the solution is

$$x = \begin{bmatrix} x \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(where $x_2, x_4$ are free)
So \( \text{Nul}(A) = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} : x_2, x_4 \text{ real} \right\} \)

\[
= \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}
\]

Notice that these vectors are linearly independent. This is no accident. Whenever \( A\mathbf{x} = \mathbf{0} \) has nontrivial solutions (that is, whenever there are free variables) and you write the solution is parametric vector form using the free variables as the parameters, then it is always true that the vectorsthat appear in the solution will be linearly independent. This is because each weight (free variable) is one of the entries in the sum, \( \mathbf{x} \), and therefore all weights must be 0 to make \( \mathbf{x} = \mathbf{0} \). Above, for example, the only way

\[
\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} * \\ x_2 \\ * \\ x_4 \end{bmatrix} = \mathbf{0}
\]

can happen is if \( x_2 = x_4 = 0 \).

Therefore when \( \text{Nul} A \neq \{\mathbf{0}\} \), the method illustrated to describe the \( \text{Nul} A \) always gives you a linearly independent set of vectors whose span is \( \text{Nul} A \)

3) Describe \( \text{Col} A : \text{Col} A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 2 \\ 4 \end{bmatrix} \right\} \). Every vector in \( \text{Col} A \) is a linear combination of these. (As it turns out, this particular set of vectors is not linearly independent: at least one, maybe more, of these vectors can be written as a linear combination of the others. So this may be an “inefficient” description of \( \text{Col} A \). We will discuss very soon how to find an “efficient” (that is, linearly independent) set of vectors that span \( \text{Col} A \).)

4) A vector \( \mathbf{b} \) is in \( \text{Col} A \) if and only if \( \mathbf{b} \) is a linear combination of the columns of \( A \), that is, if \( A\mathbf{x} = \mathbf{b} \) has a solution. For example, to determine whether \( \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \\ 4 \end{bmatrix} \) is in \( \text{col} A \), you would
need to check whether $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has a solution by row reducing the augmented matrix. If you do that, you find that the equation is inconsistent, so \( \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \) is not in col \( A \).