An Economic Example

The example illustrates how a system of linear equations might describe the functioning (production and consumption) in a very simple economy. The thing to focus on here is not the numbers but how the equations are set up to “describe” a situation.

Suppose an economy has only 4 sectors: Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). In this simple example, all goods are bought and sold back and forth among these sectors; it is a “closed exchange economy.”

We set up a matrix to exhibit the back-and-forth of goods (measured in $ value) between the sectors: an exchange matrix.

\[
\begin{bmatrix}
A & E & M & T \\
.65 & .30 & .30 & .20 \\
.10 & .10 & .15 & .10 \\
.25 & .35 & .15 & .30 \\
0 & .25 & .40 & .40
\end{bmatrix}
\rightarrow \begin{bmatrix}
A \\
E \\
M \\
T
\end{bmatrix}
\]

So, for example, 35% (or 35 cents of each $1 of goods) produced by the energy sector is consumed the manufacturing sector, and 10% of the production of energy is consumed by the energy sector (to make more energy). Notice that the total production of each section (sum of a column) is 100% (or $1).

Suppose \( p_A, p_E, p_M, p_T \) represent the total production of goods (\( in \infty \)) by in each sector.

Then to “cost” to sector \( A \) for what it consumes is

\[.65P_A + .30P_E + .30P_M + .20P_T\]

This expense must be “paid for” using the value of sector \( A \)'s goods, \( P_A \), so

\[.65P_A + .30P_E + .30P_M + .20P_T = P_A.\]

Similarly, we need

\[.10P_A + .10P_E + .15P_M + .10P_T = P_E\]
\[.25P_A + .35P_E + .15P_M + .30P_T = P_M\]
\[.25P_E + .40P_M + .40P_T = P_T\]
Rearranging the equations gives the linear system

\[
\begin{align*}
-0.35P_A + 0.30P_E + 0.30P_M + 0.20P_T &= 0 \\
0.10P_A - 0.90P_E + 0.15P_M + 0.10P_T &= 0 \\
0.25P_A + 0.35P_E - 0.85P_M + 0.30P_T &= 0 \\
0.25P_E + 0.40P_M - 0.60P_T &= 0
\end{align*}
\]

The augmented matrix is

\[
\begin{bmatrix}
-0.35 & 0.30 & 0.30 & 0.20 & 0 \\
0.10 & -0.90 & 0.15 & 0.10 & 0 \\
0.25 & 0.35 & -0.85 & 0.30 & 0 \\
0 & 0.25 & 0.40 & -0.60 & 0
\end{bmatrix}
\]

The row reduced echelon form (with entries rounded to 2 decimal places for convenience) is

\[
\begin{bmatrix}
1 & 0 & 0 & -2.03 & 0 \\
0 & 1 & 0 & -0.53 & 0 \\
0 & 0 & 1 & -1.17 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Writing \( \approx \) instead of \( = \) (since we rounded decimals), we have

\[
\begin{align*}
P_A & \approx 2.03P_T \\
P_E & \approx 0.53P_T \\
P_M & \approx 1.17P_T \\
P_T & \text{ is free}
\end{align*}
\]

What does this mean?

if a value for the free variable \( P_T \) = a total production level \( P_T \) (\$) is chosen

then the production levels for the other sectors can be set so that all the equations (*) are satisfied. In other words we have found a “equilibrium value” for the economy where each sector is getting exactly what it needs and is producing exactly what's necessary to pay for what's needed. Everybody's happy!

We will look at a slightly more sophisticated version of the model (called an “open economy”) later.