Right and Left Inverses for a Matrix

$D$ is called a **right inverse** for a $m \times n$ matrix $A$ if $AD = I_m$ (so $D$ must be $n \times m$). For example, if $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 4 \end{bmatrix}$, then a right inverse for $A$ is $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ because $AD = \begin{bmatrix} 1 & 0 \end{bmatrix} = I_2$.

But if $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AE = \begin{bmatrix} 1 & 0 \end{bmatrix} = I_2$ also, so $E$ is another right inverse for $A$.

If $A$ has a right inverse, it is not necessarily unique.

$C$ is called a **left inverse** for a $m \times n$ matrix $A$ if $CA = I_n$ (so $C$ must be $n \times m$).

It turns out that the matrix $A$ above has no left inverse (see below). This is no accident! The following theorem says that if $A$ has both a right and a left inverse, then $A$ must be square.

**Theorem** If $A$ is $m \times n$ and if

i) $D$ is a a right inverse for $A$ (so $AD = I_m$) and

ii) $C$ is a left inverse for $A$ (so $CA = I_n$)

then $m = n$ (so $A$ is square). Moreover, $A$ is invertible and $A^{-1} = C = D$.

**Proof** Suppose $A$ is $m \times n$.

If $AD = I_m$, then the equation $Ax = b$ has a solution for every possible $b$ in $\mathbb{R}^m$ (given a $b$, just let $x = Db$; then $Ax = A(Db) = I_m b = b$.

Therefore $A$ has a pivot position in every row. This forces $m \leq n$, since every pivot position must be in a different column.

If $CA = I_n$, consider the equation $Ax = 0$. Then $CAx = C0 = 0$. But $CAx = I_n x = x$, so $x = 0$. In other words, $Ax = 0$ has a unique solution and therefore the columns of $A$ must be linearly independent and therefore each column must be a pivot column. Since each pivot position must be in a different row, this forces $n \leq m$. 
So, combining the two paragraphs gives that \( m = n \). Since \( A \) is now known to be square, the Invertible Matrix Theorem says that \( A \) is invertible and that \( C = D = A^{-1} \).