One way not to write a proof

Theorem If a point \( x \) in \( (X, T) \) is not isolated, then every open set that contains \( x \) is infinite.

Proof Assume not: suppose \( x \) is not isolated and that there is a finite open set that contains \( x \).

Since \( x \) is not isolated, we can choose distinct points \( x_n \) such that \( (x_n) \to x \).

(... details about why omitted for purposes of this example...)

If \( O \) is any open set that contains \( x \), the sequence \( (x_n) \) is eventually in \( O \), so every open set \( O \) that contains \( x \) is infinite.

This contradicts our assumption that there is a finite open set that contains \( x \).

Therefore the theorem is true. ●

The proof (after filling in a couple of missing details) is logically correct — but the logic, at best, is unnecessarily confused. Here's an analysis of the logic in the preceding proof, beginning with labels on some of the same parts.

**Theorem** If \( x \) is not isolated in \( (X, T) \), then every open set that contains \( x \) is infinite.

**Proof** Assume not: that \( x \) is not isolated and there is a finite open set that contains \( x \).

Since \( x \) is not isolated, we can choose distinct points \( x_n \) such that \( (x_n) \to x \).

(...some details about why omitted for this example...)

If \( O \) is any open set that contains \( x \), the sequence \( (x_n) \) is eventually in \( O \), so every open set \( O \) that contains \( x \) is infinite.

Since we assumed (\( \sim Q \)) that there is a finite open set that contains \( x \), we have contradicted our assumption.

Therefore the theorem is true. ●
The proof is presented as a “proof by contradiction.” But notice that the argument in the box, by itself, is a complete direct proof of the theorem. The boxed argument has logical form:

Assume $P$

Argue that $P \Rightarrow Q$

Therefore $Q$ (as desired).

In the long version, the opening assumption $\sim Q$ is never actually used in the rest of the argument. It is there simply as a “straw man” to be contradicted at the end.

The complicated logic of the longer version is:

Assume $P$ and $\sim Q$.

Argue that $P \Rightarrow Q$ (direct proof, not using assumption $\sim Q$)

Therefore $Q$

But this contradicts the assumption $\sim Q$

Since we got a contradiction, we conclude $Q$.

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A “genuine” proof by contradiction would assume $P$ and $\sim Q$ and use both assumptions to derive a contradiction of some known previous known result:

For example: Assume $P$ and $\sim Q$.

(Argument using both of these assumptions) ..., so $\sqrt{2}$ is rational.

But this is impossible, so our assumption was wrong.