Homework 3 is due in class on Thursday, February 18.

1. A space \((X, T)\) is called a \(T_0\) space if whenever \(x \neq y \in X\), then \(N_x \neq N_y\) (equivalently, either there is an open set \(U\) containing \(x\) but not \(y\), or vice-versa). Notice that the \(T_0\) condition is weaker than \(T_1\) (see example III.2.6.4). Clearly, a subspace of a \(T_0\)-space is \(T_0\).

   a) Prove that a nonempty product \(X = \prod \{X_\alpha : \alpha \in A\}\) is \(T_0\) iff each \(X_\alpha\) is \(T_0\).

   b) Let \(S\) be “Sierpinski space” — that is, \(S = \{0, 1\}\) with the topology \(T = \{\emptyset, \{1\}, \{0, 1\}\}\). Use the embedding theorems to prove that a space \(X\) is \(T_0\) iff \(X\) is homeomorphic to a subspace of \(S^m\) for some cardinal \(m\). (Nearly all interesting spaces are \(T_0\), so nearly all interesting spaces are (topologically) just subspaces of \(S^m\) for some cardinal \(m\).)

   Hint: \(\Rightarrow\) : for each open set \(U\) in \(X\), let \(\chi_U\) be the characteristic function of \(U\). Use an embedding theorem.

2. a) Let \((X, d)\) be a metric space. Prove that \(C(X)\) (= the collection of continuous real-valued functions on \(X\)) separates points from closed sets. (Since \(X\) is \(T_1\), \(C(X)\) therefore also separates points).

   b) Suppose \(X\) is any \(T_0\) topological space for which \(C(X)\) separates points and closed sets. Prove that \(X\) can be embedded in a product of copies of \(\mathbb{R}\).

3. A space \(X\) is called totally disconnected every connected subset \(A\) satisfies \(|A| \leq 1\) (equivalently, if all components in \(X\) are singletons). Prove that a totally disconnected compact Hausdorff space is homeomorphic to a closed subspace of \(\{0, 1\}^m\) for some \(m\).

   Hint: see Lemma V.5.6.

4. a) Let \(\sim\) be the equivalence relation on \(\mathbb{R}^2\) given by \((x_1, y_1) \sim (x_2, y_2)\) iff \(y_1 = y_2\). Prove that \(\mathbb{R}^2/\sim\) is homeomorphic to \(\mathbb{R}\).

b) Find a counterexample to the following assertion: if \(\sim\) is an equivalence relation on a space \(X\) and each equivalence class is homeomorphic to the same space \(Y\), then \((X/\sim) \times Y\) is homeomorphic to \(X\).

   Why might someone ever wonder whether this assertion might be true? In part a), we have \(X = \mathbb{R} \times \mathbb{R}\), each equivalence class is homeomorphic to \(\mathbb{R}\) and \((X/\sim) \times Y \simeq \mathbb{R} \times \mathbb{R} \simeq X\). In this example, you “divide out” equivalence classes that all look like \(\mathbb{R}\), then “multiply” by \(\mathbb{R}\), and you’re back where you started.

c) Let \(g : \mathbb{R}^2 \to \mathbb{R}\) be given by \(g(x, y) = x^2 + y^2\). Then the quotient space \(\mathbb{R}^2/g\) is homeomorphic to what familiar space?
5. For $x, y \in [0, 1]$, define $x \sim y$ iff $x - y$ is rational. Prove that the corresponding quotient space $[0, 1]/\sim$ is trivial.

6. Suppose $X_s (s \in S)$ and $Y_t (t \in T)$ are pairwise disjoint spaces. Prove that $\sum_{s \in S} X_s \times \sum_{t \in T} Y_t$ is homeomorphic to $\sum_{s \in S, t \in T} (X_s \times Y_t)$.

7. A base for the closed sets in $(X, T)$ is a collection of $F$ of closed sets such that every closed set $F$ is an intersection of sets from $F$. Clearly, $F$ is a base for the closed sets in $X$ iff $B = \{O : O = X - F, F \in F\}$ is a base for the open sets in $X$.

For a polynomial $P$ in $n$ real variable, define the zero set of $P$ as
\[
Z(P) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : P(x_1, x_2, \ldots, x_n) = 0\}
\]

a) Prove that $\{Z(P) : P$ a polynomial in $n$ real variables$\}$ is the base for the closed sets of a topology (called the Zariski topology) on $\mathbb{R}^n$.

b) Prove that the Zariski topology on $\mathbb{R}^n$ is $T_1$ but not $T_2$.

c) Prove that the Zariski topology on $\mathbb{R}$ is the cofinite topology, but that if $n > 1$, the Zariski topology on $\mathbb{R}^n$ is not the cofinite topology.

Note: The Zariski topology arises in studying algebraic geometry. After all, the sets $Z(P)$ are rather special geometric objects—those “surfaces” in $\mathbb{R}^n$ which can be described by polynomial equations $P(x_1, x_2, \ldots, x_n) = 0$. 