Homework 5 is due in class on Thursday, March 18.

1. Let $(X, d)$ be a metric space and $S \subseteq X$. Prove that if each continuous $f : S \to \mathbb{R}$ extends to a continuous $g : X \to \mathbb{R}$, then $S$ is closed. (*The converse, of course, is from Tietze’s Extension Theorem.*)

2. Prove that a Hausdorff space $X$ is normal iff for each finite open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of $X$, there exist continuous functions $f_i : X \to [0, 1]$ ($i = 1, \ldots, n$) such that $\sum_{i=1}^{n} f_i(x) = 1$ for each $x \in X$ and such that, for each $i$, $f_i | X - U_i = 0$. (*Such a set of functions is called a partition of unity subordinate to the finite cover $\mathcal{U}$.*)

3. Suppose $X$ is a $T_1$ space. $X$ is called perfectly normal if whenever $A$ and $B$ are disjoint nonempty closed sets in $X$, there is an $f \in C(X)$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$.
   a) Prove that every metric space $(X, d)$ is perfectly normal.
   b) Prove that $X$ is perfectly normal iff $X$ is $T_1$ and every closed set in $X$ is a $G_δ$-set.

4. A space $X$ is called locally compact if each $x \in X$ has a neighborhood base consisting of compact neighborhoods. For example: every discrete space is locally compact, $\mathbb{R}^n$ is locally compact, and $\mathbb{Q}$ is not locally compact.
   a) Prove that a compact Hausdorff space is locally compact.

Suppose $X$ is a locally compact Hausdorff space that is not compact. For $x \in X$, let $B_x$ be a neighborhood base at $x$ consisting of compact neighborhoods. Let $p$ be a point not in $X$, and define $X^* = X \cup \{p\}$. Put a topology on $X^*$ by using the following definition for neighborhood bases:

\[
\begin{cases}
B_x, & \text{for } x \in X \\
B_p = \{B \subseteq X^* : p \in B \text{ and } X^* - B \text{ is compact} \}
\end{cases}
\]

(*You can assume that this definition satisfies the conditions in the Neighborhood Base Theorem III.5.2.*)

b) Prove that $X^*$ is a compact Hausdorff space and that $X$ is dense in $X^*$.

c) Suppose $X$ is a noncompact Hausdorff space, but not locally compact and we construct $X^*$ using exactly the same definition. In that case, which statements in b) are no longer necessarily true? (*Aside: ask yourself the same questions if the definition for $X^*$ is applied to a compact Hausdorff space $X$; or to a locally compact noncompact $X$ that is not Hausdorff.*

(OVER →)
d) Suppose again that $X$ is a locally compact, noncompact Hausdorff space, that $q \notin X$ and that $Y = X \cup \{q\}$ is a compact Hausdorff space with $X$ as a dense subspace. Define $h : X^* \to Y$ by $h(p) = q$ and $h(z) = z$ for $z \in X$. Prove that $h$ is a homeomorphism.

e) Suppose $X = (0, 1)$ and we construct $X^*$. What familiar space is $X^*$?
   Note: the answer is the same if $X = \mathbb{R}$ since $(0, 1)$ and $\mathbb{R}$ are homeomorphic: this follows easily from the argument in d).