Why is the Evaluation Theorem (Fundamental Theorem of Calculus, Part II) True?

First, we want to recall the Mean Value Theorem (MVT). It states that if \( F(x) \) is differentiable on an interval \([u, v]\), then there must be a point \( c \) between \( u \) and \( v \) where

\[
\frac{F(v) - F(u)}{v - u} = F'(c),
\]

or, rewritten, where \( F(v) - F(u) = F'(c)(v - u) \).

Suppose that \( f \) is continuous on \([a, b]\) and that \( F \) is an antiderivative for \( f \).

The “recipe” for \( \int_a^b f(x) \, dx \): we subdivide \([a, b]\) into \( n \) equal parts of length \( \Delta x \), choose sample points \( x_i^* \) in each subinterval (how we do this is irrelevant) and form the Riemann sum \( \sum_{i=1}^n f(x_i^*) \Delta x \). Then \( \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*) \Delta x \). We will follow the recipe and we'll use our freedom to choose the \( x_i^* \)'s in a very clever way.

The subintervals are \([a = x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots [x_{n-2}, x_{n-1}], [x_{n-1}, x_n = b] \).

Write \( F(b) - F(a) = F(x_n) - F(x_0) \)

\[
= (F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + \ldots + (F(x_1) - F(x_{i-1})) + \ldots + (F(x_2) - F(x_1)) + (F(x_1) - F(x_0))
\]

\[
= \sum_{i=1}^n F(x_i) - F(x_{i-1}).
\]

Now use the MVT on each term: on \([x_{i-1}, x_i]\), \( F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) \) for some \( c_i \) in the subinterval. But \( F'(c_i)(x_i - x_{i-1}) = f(c_i) \Delta x \), since \( F' = f \).
So we have \( F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} f(c_i) \Delta x. \)

We now rename \( c_i \) as \( x_i^* \). Then \( F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*) \Delta x \), a Riemann sum where we allowed the MVT, on our behalf, to choose the sample points in each subinterval.

So, for each \( n \), we can construct a Riemann sum in just this way. Since it doesn't matter in the definition how the \( x_i^* \)'s got chosen, we know that

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \int_{a}^{b} f(x) \, dx
\]

But, for each \( n \), we constructed the Riemann sums so that for each one,

\[
F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*) \Delta x.
\]

Therefore \( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \lim_{n \to \infty} F(b) - F(a) = F(b) - F(a) \) (since \( F(b) - F(a) \) is a constant).

We conclude that \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \), which is what we wanted to prove.