Math 131
Exam 2 Solutions, Fall 2004

Part I consists of 14 multiple choice questions (worth 5 points each) and 5 true/false question (worth 1 point each), for a total of 75 points. Mark the correct answer on the answer card. For Part I, only the answer on the card will be graded.

1. If \( y = \frac{x^3}{64} + \sqrt[3]{x} \), what is \( \frac{dy}{dx} \bigg|_{x=8} \)?

A) \( \frac{4}{3} \)  
B) \( \frac{37}{12} \)  
C) \( \frac{40}{8} \)  
D) \( \frac{69}{61} \)  
E) \( \frac{75}{12} \)  
F) \( \frac{3}{8} \)  
G) \( \frac{7}{12} \)  
H) \( \frac{35}{8} \)  
I) \( \frac{127}{64} \)  
J) \( \frac{14}{3} \)

**Solution:**

\[ y = \frac{1}{64}x^3 + x^{\frac{1}{3}}, \text{ so } \frac{dy}{dx} = \frac{3}{64}x^2 + \frac{1}{3}x^{-\frac{2}{3}} = \frac{3}{64}x^2 + \frac{1}{3\sqrt[3]{x^2}}. \text{ Therefore} \]

\[ \frac{dy}{dx} \bigg|_{x=8} = \frac{3}{64}(8^2) + \frac{1}{3\sqrt[3]{64}} = 3 + \frac{1}{12} = \frac{37}{12} \]

2. The slope of the tangent line to \( f(t) = \frac{a + e^t}{e^{3t}} \) where \( t = 0 \) is 5. What is \( a \)?

A) 0  
B) -1  
C) -2  
D) -3  
E) -4  
F) 5  
G) 4  
H) 3  
I) 2  
J) 1

**Solution:**

Using the quotient rule gives \( f'(t) = \frac{e^{3t}(e^t) - (a + e^t)(2e^t)}{(e^{3t})^2} \).

Therefore \( 5 = f'(0) = \frac{1 - (a+1)(2)}{1} = -2a - 1 \), so \(-2a = 6\), \(a = -3\).
3. Suppose $F(x) = f(g(x))$ and that:

\[
\begin{align*}
g(2) &= 6 \\
g'(2) &= 3 \\
f'(2) &= -1 \\
f(2) &= 7 \\
f'(6) &= -4
\end{align*}
\]

what is $F'(2)$?

A) 0     B) 2     C) 21     D) 42     E) -24
F) 4     G) 18     H) -6     I) -3     J) -12

By the chain rule: $F'(x) = (f(g(x)))' = f'(g(x))g'(x)$.

So $F'(2) = (f(g(x)))'(2) = f'(g(2))g'(2)$

$= f'(6)g'(2) = (-4)(3) = -12$.

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4. If $g(4) = 2$ and $g'(4) = 3$, what is $(\frac{\sqrt{x}}{g(x)})'(4)$?

A) $\frac{1}{2}$     B) $-\frac{4}{3}$     C) $\frac{14}{9}$     D) $-\frac{13}{4}$     E) $-\frac{11}{8}$
F) $\frac{33}{2}$     G) -4     H) $\frac{17}{6}$     I) $-\frac{17}{3}$     J) 3

Using the quotient rule gives $(\frac{\sqrt{x}}{g(x)})' = \frac{g'(x)\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{g(x)}g'(x)}{(g(x))^2}$

$= \frac{g(4)\cdot\frac{1}{2\sqrt{4}} - \frac{\sqrt{4}}{g(4)}g'(4)}{(g(4))^2}$, so $(\frac{\sqrt{x}}{g(x)})'(4) = \frac{g(4)\cdot\frac{1}{2\sqrt{4}} - \frac{\sqrt{4}}{g(4)}g'(4)}{(g(4))^2}$

$= 2(\frac{1}{2}) - 2(3) = \frac{1}{2} - 6 = -\frac{11}{2} = -\frac{11}{4} = -\frac{11}{8}$. 
5. The figure shows the graph of two functions $f$ and $g$.

Let $h(x) = f(x)g(x)$. What is $h'(2)$?

A) $\frac{1}{2}$  B) $\frac{-4}{3}$  C) $\frac{14}{9}$  D) $\frac{-13}{4}$  E) $\frac{-11}{6}$

F) $\frac{33}{2}$  G) $-4$  H) $\frac{17}{6}$  I) $-\frac{17}{3}$  J) $3$

From the graph, $f(2) = 2$ and $g(2) = -1$. Computing the slopes of the appropriate straight line segments, we get $f'(2) = \frac{1}{2}$ and $g'(2) = -\frac{2}{3}$. By the product rule,

$$h'(2) = f(2)g'(2) + g(2)f'(2) = (2)\left(-\frac{2}{3}\right) + (-1)\left(\frac{1}{2}\right) = -\frac{4}{3} - \frac{1}{2} = -\frac{11}{6}.$$ 

6. At time $t$ (hrs), the size $P$ of a certain population of bacteria is $P = 5^{t^2+t}$.

How fast is $P$ changing at time $t = 1$? (Round your answer to the nearest integer. All answers have the units “bacteria/hr.”)

A) 23  B) 130  C) 25  D) 18  E) 137

F) 143  G) 121  H) 97  I) 243  J) 87

$$\frac{dP}{dt} = (\ln 5)5^{t^2+t}(2t+1).$$

At $t = 1$, the rate of change of $P$ is

$$\left.\frac{dP}{dt}\right|_{t=1} = (\ln 5)(5^2)(3) = (\ln 5)(75) \approx 120.71 \approx 121 \text{ (bacteria/hr)}.$$
7. A point is moving along a straight line. At time \( t \) its velocity \( v(t) = t^2 - 3t + 2 \).

Exactly two of the following statements are true. Which ones are true?

i) The point is moving in the positive direction for times \( t < 1 \).
ii) The graph of the position function \( s = f(t) \) is always concave up.
iii) During the times \( 8 < t < 10 \), the point is speeding up.
iv) When \( \frac{ds}{dt} \) is increasing, the point must be moving in the positive direction.
v) With the information given, it is possible to calculate the position at time \( t = 0 \).

A) i, ii  B) i, iii  C) i, iv  D) i, v  E) ii, iii 
F) ii, i  G) ii, v  H) iii, iv  I) iii, v  J) iv, v

\[
v = t^2 - 3t + 2 = (t - 1)(t - 2).
\]
If \( t < 1 \), \( v > 0 \), so i) is true.

\[
\frac{d^2s}{dt^2} = \frac{dv}{dt} = 2t - 3 > 0 \text{ when } t > \frac{3}{2}, \text{ so } f(t) \text{ is concave up only when } t > \frac{3}{2}.
\]
Therefore ii) is false.

For times \( 8 < t < 10 \), \( v > 0 \) and \( a > 0 \), so \( v \) is positive and increasing. This means the point is speeding up, so iii) is true.

If \( v = \frac{ds}{dt} \) is increasing, that means \( \frac{d^2s}{dt^2} = \frac{dv}{dt} = a > 0 \), so \( s = f(t) \) is concave up. But \( f(t) \) could nevertheless be decreasing. Therefore iv) is false.

v) is false because just from the velocity \( v = t^2 - 3t + 2 \), it is impossible to compute \( s = f(t) \). For example, we can’t tell whether \( s = f(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t \) or \( s = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + 137 \): both of these possible position functions have the same velocity function. (If you know Tom’s driving velocity at each time \( t \), is that enough to figure out where he started? No.)
8. If $g(x) = \sec^2(2x)$, what is $g'(\frac{\pi}{6})$?

A) 0  B) 1  C) 2  D) $\frac{1}{2}$  E) $\sqrt{2}$

F) $\frac{1}{\sqrt{2}}$  G) 8  H) $\frac{1}{5}$  I) $\frac{1}{2}$  J) $\frac{2}{\sqrt{3}}$

\[ g(x) = \sec^2(2x) = (\sec(2x))^2. \]

Using the chain rule we get
\[ g'(x) = 2(\sec(2x))^1 \cdot \frac{d}{dx}(\sec(2x)) = 2\sec(2x) \cdot \sec(2x) \cdot \tan(2x) \cdot \frac{d}{dx}(2x) \]
\[ = 2\sec(2x) \cdot \sec(2x) \cdot \tan(2x) \cdot (2) = 4\sec^2(2x) \cdot \tan(2x). \]

So
\[ g'(\frac{\pi}{6}) = 4\sec^2(\frac{\pi}{4}) \tan(\frac{\pi}{4}) = 4(\sqrt{2})^2(1) = 8. \]

9. Suppose $f(3) = 1$ and $f'(3) = 2$. What is the estimated value, using linear approximation, for $f(2.99)$?

A) 0.93  B) 0.94  C) 0.95  D) 0.96  E) 0.97

F) 0.98  G) 0.99  H) 3  I) 1.02  J) 1.03

The linear approximation at 3 is an approximation for the values of $f(x)$ near 3. It is
\[ L(x) = f(3) + f'(3)(x - 3) = 1 + 2(x - 3). \]

Therefore
\[ f(2.99) \approx L(2.99) = 1 + 2(2.99 - 3) = 1 + 2(-0.01) = 0.98. \]

10. The cost ($) of producing $x$ toasters at the G.W. Crumbly Factory is
\[ C(x) = 1000000 - 0.0001(x - 90000)^2. \]

What is the marginal cost when $x = 30000$?
(Round your answer to the nearest cent.)

A) 12.87  B) 12.97  C) 14.32  D) 13.00  E) 15.67

F) 13.69  G) 18.67  H) 11.43  I) 12.00  J) 10.45

The marginal cost is $C'(30000)$. Since $C'(x) = -0.0002(x - 90000)$,
\[ C'(30000) = -0.0002(-60000) = 12 \text{ ($\$)}.
\]

Note: $C'(30000)$ is called the marginal cost because it approximates $C(30001) - C(30000) = \text{the cost of manufacturing coaster #30 001.}$
11. The figure below shows the graph of the derivative $y = f'(x)$ for some function $y = f(x)$.

Exactly two of the following statements are true. Which ones are true?

i) $f(x)$ has an inflection point at $x = 3$
ii) $f(x)$ has a local maximum at $x = 4$
iii) $f''(x)$ is increasing for $5 < x < 6$
iv) $f(x)$ is concave down for $3 < x < 5$
v) $f''(x) < 0$ for $1 < x < 3$

A) i, ii B) i, iii C) i, iv D) i, v E) ii, iii F) ii, iv G) ii, v H) iii, iv I) iii, v J) iv, v
At $x = 3$, the derivative $f''(x)$ switches from decreasing to increasing so at $x = 3$ $f'''(x)$ changes from negative to positive. Therefore $f$ switches concavity at $x = 3$. So i) is true.

Just to the left of $x = 4$, $f'(x) < 0$ and just to the right of $x = 4$, $f'(x) > 0$. So just to the left of $x = 4$, $f(x)$ is decreasing and just to the right of $x = 4$, $f(x)$ is increasing. So $f$ has a local minimum at $x = 4$. Therefore ii) is false.

If $f''(x)$ were increasing for $5 < x < 6$, then $f'(x)$ would be concave up on that interval, but it isn't. So iii) is false.

Since $f'(x)$ is increasing for $3 < x < 5$ so $f''(x) > 0$ on that interval and $f(x)$ is concave up. So iv) is false.

v) For $1 < x < 3$, $f'(x)$ is decreasing so $f''(x) < 0$. Therefore v) is true.

12. Let $y = xe^{-8x^2}$. What is the smallest value of $a$ that makes the statement “$y$ is decreasing for $x > a$” true?

A) 8  B) 4  C) 2  D) $\frac{1}{2}$  E) $\frac{1}{4}$
F) 0  G) $-\frac{1}{4}$  H) $-\frac{1}{2}$  I) $-1$  J) $-2$

$f(x)$ is decreasing where $f'(x) < 0$.

$f'(x) = e^{-8x^2}(1) + x(-16xe^{-8x^2}) = e^{-8x^2} - 16x^2e^{-8x^2}$

$= e^{-8x^2}(1 - 16x^2)$.

Since $e^{-8x^2} > 0$, $f'(x) < 0$ exactly where $1 - 16x^2 < 0$, that is when $16x^2 > 1$. Solving this inequality gives $x < -\frac{1}{4}$ or $x > \frac{1}{4}$.

The statement “$y$ is decreasing if $x > \frac{1}{4}$” is true.
13. What is \( \lim_{h \to 0} \frac{(1+h)^{10000} - 1}{h} \) ?

A) 0  B) 10000  C) \( e \)  D) 100  E) 1
F) \( \frac{1}{e} \)  G) \( \frac{1}{100} \)  H) \( \frac{1}{10000} \)  I) 2\(^{10000} \)  J) \( \infty \) (limit d.n.e.)

If we let \( f(x) = x^{10000} \), then \( f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^{10000} - 1}{h} \).

But we know now that \( f'(x) = 10000x^{9999} \), so \( f'(1) = 10000 \).

14. There is one (and only one) line through the point \((2,1)\) that is tangent to the graph of \( y = \frac{x}{x-1} \) at some point \( P \) on the graph. What is the \( x \)-coordinate of \( P \)?

A) \( \frac{3}{2} \)  B) 1  C) 2  D) -1  E) -2
F) \( \frac{1}{4} \)  G) \( -\frac{2}{3} \)  H) \( \frac{5}{3} \)  I) 3  J) \( \frac{3}{4} \)

\( f'(x) = \frac{(x-1)(1-x)}{(x-1)^2} = \frac{-1}{(x-1)^2}. \)

If \( (a, \frac{a}{a-1}) \) is a point on the graph, then the slope at this point is \( \frac{-1}{(a-1)^2} \). Therefore the tangent line has equation \( y - \frac{a}{a-1} = \frac{-1}{(a-1)^2} (x - a) \).

If \((2,1)\) is going to be on the tangent line, it has to satisfy the equation of the tangent line, so

\[ 1 - \frac{a}{a-1} = \frac{-1}{(a-1)^2} (2 - a). \]

Multiplying both sides by \((a-1)^2\) gives

\[ (a-1)^2 - a(a-1) = -1(2 - a), \text{ or} \]
\[ a^2 - 2a + 1 - a^2 + a = -2 + a, \text{ or} \]
\[ 2a = 3 \]
\[ a = \frac{3}{2} \]

Questions 15-19 are true/false questions.

15. If \( \lim_{h \to 0} f(2+h) = f(2) \), then \( f \) must have a derivative at 2.

A) True  B) False
\[ \lim_{{h \to 0}} f(2 + h) = f(2) \] simply states that \( f \) is continuous at 2. (If you substitute \( x = 2 + h \), then \( \lim_{{h \to 0}} f(2 + h) = \lim_{{x \to 2}} f(x) = f(2) \).) But a function that is continuous at 2 does not have to be differentiable at 2—consider for example, \( f(x) = |x - 2| \).

16. There is one and only one point where the function \( f(x) = |x^2 + 1| + |x - 2| \) fails to have a derivative.

   A) True   B) False

Since \( x^2 + 1 > 0 \), \( f(x) = |x^2 + 1| + |x - 2| = x^2 + 1 + |x - 2| \). This function fails to have a derivative only at \( x = 2 \).

17. Suppose \( T, P, V \) (the temperature, pressure, and volume of a gas in a container) are related by \( \frac{PV}{T} = k \), where \( k \) is a constant. If the temperature is held constant, then \( \frac{dT}{dP} = -\frac{kT}{P^2} \).

   A) True   B) False

Solving \( \frac{PV}{T} = k \) for \( V \) gives \( V = \frac{kT}{P} \). If \( T \) is held constant, then \( V = kT \frac{1}{P} \) is a function of \( P \) alone, and \( \frac{dV}{dP} = kT (-1P^{-2}) = -\frac{kT}{P^2} \).

18. Suppose \( f'(x) = (x - 2)^2(x - 5) \). Then \( f(x) \) has either a local maximum or a local minimum at \( x = 2 \).

   A) True   B) False

Near \( x = 2 \) (on the left or the right), \( (x - 5) < 0 \). Furthermore, near \( x = 2 \) (on the left or the right), \( (x - 2)^2 > 0 \). Therefore near \( x = 2 \) (on the left or the right), \( f'(x) < 0 \). This means the graph of \( f(x) \) decreases as \( x \to 2^- \), flattens out for a horizontal tangent \( x = 2 \), and then continues to decrease just to the right of \( x = 2 \). There is neither a local max nor min at \( x = 2 \).

19. If \( f \) has a derivative at 3, then \( \lim_{{x \to 3}} f(x) = f(3) \).

   A) True   B) False
If $f$ has a derivative at 3, then $f$ must be continuous at 3, and that's just what
\[ \lim_{x \to 3} f(x) = f(3) \] means.
Part II: (25 points) In each problem, clearly show your solution in the space provided. “Show your solution” does not simply mean “show your scratch work”—you should cross out any scratch work that turned out to be wrong or irrelevant and, where appropriate, present a readable, orderly sequence of steps showing how you got the answer. Generally, a correct answer without supporting work may not receive full credit.

20. For each function $y = f(x)$ given below, find the derivative. After all the differentiation is completed, no further simplifications are necessary. For example, an answer that looking like

\[
\frac{dy}{dx} = \frac{2(3\cos(2x) - 4x(2x + 1)(3)}{(x + 1)(x + 3) - (2)(3)x}
\]

would require no further simplification.

a) $f(x) = \frac{(3x - 4)^{10}}{\cos x}$

\[
f'(x) = \frac{(\cos x)10(3x - 4)^9(3) - (3x - 4)^{10}(-\sin x)}{\cos^2 x} = \frac{(3x - 4)^9(30\cos x + (\sin x)(3x - 4))}{\cos^2 x}
\]

(this is a sufficient answer)

b) $f(x) = 5\sec(\sqrt{x})$

\[
f'(x) = (\ln 5) \cdot 5\sec(\sqrt{x}) \cdot \sec(\sqrt{x}) \cdot \frac{d}{dx}(\sec(\sqrt{x}))
= (\ln 5) \cdot 5\sec(\sqrt{x}) \cdot \sec(\sqrt{x}) \cdot \tan(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x})
= (\ln 5) \cdot 5\sec(\sqrt{x}) \cdot \sec(\sqrt{x}) \cdot \tan(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}
\]

c) $f(x) = \tan(\sin(e^{3x^2}))$

\[
f'(x) = \sec^2(\sin(e^{3x^2})) \cdot \frac{d}{dx}(\sin(e^{3x^2})) = \sec^2(\sin(e^{3x^2})) \cdot \cos(e^{3x^2}) \cdot \frac{d}{dx}(e^{3x^2})
= \sec^2(\sin(e^{3x^2})) \cdot \cos(e^{3x^2}) \cdot e^{3x^2} \cdot d\frac{d}{dx}(3x^2)
= \sec^2(\sin(e^{3x^2})) \cdot \cos(e^{3x^2}) \cdot e^{3x^2} \cdot 6x
\]

21. The graph of a function $y = f(x)$ is shown below. On the grid beneath it, draw a reasonable graph for $f'(x)$.

Be sure your picture clearly indicates the value of $f'(2)$ (the tangent line at $x = 2$ is drawn to help you), the places where the derivative is 0, the places where the derivative
does not exist, and where $f'(x)$ is increasing or decreasing. If those things are done, then the precise shape of $f'(x)$ is not important. If you are estimating slopes, be sure to look at the scale on each axis.

For $-2 < x < -1$, $f(x)$ has slope 1, so $f'(x) = 1$
For $-1 < x < 0$, $f(x)$ has slope $-1$, so $f'(x) = -1$
At $x = -1$, the graph of $f$ has a sharp corner, so $f'(-1)$ does not exist.

$f'(0)$ does not exist because $f$ is not continuous at 0.
Just to the right of 0, the tangent lines have a small (maybe $\approx \frac{1}{3}$) positive slope, and the slopes are positive and slopes $\to \infty$ as $x \to 1^-$ where there is a vertical tangent.
At $x = 1$, there is a vertical tangent and $f'(1)$ does not exist.

For $1 < x < 2$, the values of $f'(x)$ are positive and decreasing. At $x = 2$, we can see that the slope of the tangent line $= f'(2) = \frac{1}{3}$

For $2 < x < 3$, $f'(x)$ continues to decrease from $\frac{1}{3}$ to 0 (corresponding to a horizontal tangent at $x = 3$). Thereafter, $f'(x)$ becomes more and more negative for $3 < x < 5$.

This information is summarized in the graph below.